Generalized Jacobi Identities, Curvature Relation, Schouten's Identity, a Phase Rule and Derivation of O(q^4) Effective Lagrangian in the Presence of External Fields Directly Within Heavy Baryon Chiral Perturbation Theory

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Abstract

This is a progress report on the extension of the analysis of [1] to constructing a complete list of O(q^4) terms in the presence of external fields, and including isospin-violation directly within Heavy Baryon Chiral Perturbation Theory (HBChPT) *without having to first construct the relativistic BChPT Lagrangian and then carry out the* $\frac{1}{\mathrm{m}}$*-reduction*. In addition to a phase rule to implement all symmetries including charge conjugation invariance directly at the nonrelativistic level, generalized Jacobi identities, curvature relation (that relates the commutator of two covariant derivatives to a linear combination of the traceless and isosinglet field strengths and the commutator of two axial vector building blocks), Schouten's identity and the relationship between the antisymmetrized covariant derivative - axial-(building block) vector commutator and another traceless field strength, are used to ensure linear independence of the terms and their low energy coupling constants. We first construct O(q^4) terms for off-shell nucleons, and then perform the on-shell reduction, again within HBChPT.

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1 Introduction

Heavy Baryon Chiral Perturbation Theory (HBChPT) is a nonrelativistic (with respect to the "heavy" baryons) effective field theory used for studying meson-baryon interactions at low energies, typically below the mass of the first non-Goldstone resonance (See [2, 3, 4]). The degrees of freedom of SU(2) ($\equiv$ isospin) HBChPT (which will be considered in this paper) are the (derivatives of) pion-triplet, the nucleon fields and the external fields.

Recently, a method was developed to generate HBChPT Lagrangian ($\mathcal{L}_{\rm HBChPT}$) for off-shell nucleons *directly within HBChPT*, which as stated in [5], can prove useful when applying HBChPT to nuclear processes in which the nucleons are bound, and hence off-shell. This method has the advantage of not having to bother to start with the relativistic BChPT Lagrangian and then carry out the nonrelativistic reduction. It is thus shorter than the standard approach to HBChPT as given in [6], showed explicitly up to $\mathrm{O}(q^3)$ in [5]. In the context of off-shell nucleons, the upshot of the method developed is a phase rule (See (6),[5]) to implement charge conjugation invariance (along with Lorentz invariance, parity and hermiticity) directly within HBChPT. The phase rule, along with additional reductions from a variety of algebraic identities, was used to construct, directly within HBChPT, a complete list of off-shell $\mathrm{O}(q^3)$ terms (in the isospin-conserving approximation and in the abscence of external fields). We also showed that the on-shell limit of the list of terms obtained matches the corresponding list in [6] (in which the HBChPT Lagrangian up to $\mathrm{O}(q^3)$ was constructed starting from the relativistic BChPT Lagrangian).

For a complete and precise calculation in the single-nucleon sector to one loop, e.g., 1-loop corrections to pion production off a single (on-shell) nucleon, because of convergence problems (assocated with the amplitude "D_2" for pion production off a single nucleon), one needs to go up to $\mathrm{O}(q^4)$ (See [4, 7, 8]). An overcomplete list of the *divergent* $\mathrm{O}(q^4)$ π-nucleon interaction terms in the presence of external fields was constructed in [8], but again starting from the relativistic BChPT Lagrangian. In [1], we constructed a complete list of $\mathrm{O}(q^4)$ terms *working entirely within the framework of HBChPT* in the absence of external fields and in the isospin-conserving approximation. In this paper, we extend the list to include external fields and do not assume isospin symmetry.

Section 2 has the basics and sets up the notations. In Section 3, reductions obtained in addition to (6) due to algebraic identities such as generalized Jacobi identities, Schouten's identitiy and curvature relation, etc. are discussed. In Section 4, the complete lists of $\mathrm{O}(q^4)$ terms is given. In Section 5 we discuss the derivation of the external field-dependent on-shell $\mathrm{O}(q^4)$ Lagrangian, again within HBChPT using the techniques of [5]. Section 6 has the conclusion and discussion on comparison of the results of this paper with those of a recent paper by Fettes et al [9], in which the $\mathrm{O}(q^4)$ list in the presence of external fields is derived, but starting from the relativistic $\mathrm{O}(q^4)$ BChPT Lagrangian.

2 The Basics

The HBChPT Lagrangian is written in terms of the "upper component" H (and its hermitian adjoint $\bar{\mathrm{H}}$), exponentially parameterized matrix-valued meson fields U, $u \equiv \sqrt{U}$, baryon ("v_μ, S_ν")

and pion-field-dependent ("$\mathrm{D}_\mu, u_\nu, \chi_\pm,\ F^\pm_{\mu\nu}, v^{(s)}_{\mu\nu}$") building blocks defined below:

$$\mathrm{H} \equiv e^{\mathrm{i}mv\cdot x}\frac{1}{2}(1+\not{v})\psi, \tag{1}$$

where $\psi \equiv$ Dirac spinor and $m \equiv$ the nucleon mass;

$$\begin{aligned} &v_\mu \equiv \text{nucleon veclocity}, \\ &\mathrm{S}_\nu \equiv \frac{i}{2}\gamma^5\sigma_{\nu\rho}v^\rho \equiv \text{Pauli} - \text{Lubanski spin operator}; \end{aligned} \tag{2}$$

$$U = \exp\left(i\frac{\phi}{F_\pi}\right),\ \text{where}\ \phi \equiv \vec{\pi}\cdot\vec{\tau}, \tag{3}$$

where $\vec{\tau} \in$ nucleon isospin generators; $\mathrm{D}_\mu = \partial_\mu + \Gamma_\mu - iv^{(s)}_\mu$ where $\Gamma_\mu \equiv \frac{1}{2}\Big((u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\Big)$ ($v^{(s)}_\mu$ is the isosinglet vector field needed to generate the electromagnetic current (See [6])); $u_\mu \equiv i\Big(u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\Big)$; $\chi_\pm \equiv u^\dagger\chi u^\dagger \pm u\chi^\dagger u$, where $\chi \equiv 2B(s+ip)$,$s - \mathcal{M}$($\equiv$ quark mass matrix) and p being the external scalar and pseudo-scalar fields; $F^\pm_{\mu\nu} \equiv u^\dagger F^R_{\mu\nu}u \pm uF^L_{\mu\nu}u^\dagger$ where $F^R_{\mu\nu} \equiv \partial_{[\mu}r_{\nu]} - i.[r_\mu, u_\nu]$, and $F^L_{\mu\nu} \equiv \partial_{[\mu}l_{\nu]} - i[l_\mu, l_\nu]$ and $v^{(s)}_{\mu\nu} \equiv \partial_{[\mu}v^{(s)}_{\nu]}$ in which $r_\mu \equiv V_\mu + A_\mu,\ l_\mu \equiv V_\mu - A_\mu$, where V_μ, A_μ are external vector and axial-vector fields.

Terms of the $\mathcal{L}_{\mathrm{(H)BChPT}}$ constructed from products of building blocks will automatically be chiral invariant. Symbolically, a term in $\mathcal{L}_{\mathrm{HBChPT}}$ can be written as just a product of the building blocks to various powers (omitting $\mathrm{H}, \bar{\mathrm{H}}$ as will be done in the rest of the paper except for Section 5):

$$\mathrm{D}^m_\alpha u^n_\beta \chi^p_+ \chi^q_- v^l_\sigma (v^{(s)}_{\sigma\omega})^k (F^+_{\rho\lambda})^t (F^-_{\mu\nu})^u \mathrm{S}^{\mathrm{r}}_\kappa \equiv (\mathrm{m,n,p,q,t,u,k}) \equiv \mathrm{O}(\mathrm{q}^{\mathrm{m+n+2p+2q+2k+2t+2u}}). \tag{4}$$

A systematic path integral derivation for $\mathcal{L}_{\mathrm{HBChPT}}$ based on a paper by Mannel et al [10], starting from $\mathcal{L}_{\mathrm{BChPT}}$ was first given by Bernard et al [7]. As shown by them, after integrating out h from the generating functional, one arrives at $\mathcal{L}_{\mathrm{HBChPT}}$:

$$\mathcal{L}_{\mathrm{HBChPT}} = \bar{\mathrm{H}}\Big(\mathcal{A} + \gamma^0\mathcal{B}^\dagger\gamma^0\mathcal{C}^{-1}\mathcal{B}\Big)\mathrm{H}, \tag{5}$$

an expression in the upper components only i.e. for non-relativistic nucleons. So the terms of $\mathcal{L}_{\mathrm{HBChPT}}$ in this paper are given as operators on the H-spinors. For off-shell nucleons, $\gamma^0\mathcal{B}^\dagger\gamma^0\mathcal{C}^{-1}\mathcal{B} \in \mathcal{A}$, and hence, listing $\mathcal{A}$-type terms will suffice.

The phase rule derived in [5] can be modified to include external fields. After doing so, one gets: HBChPT terms (that are Lorentz scalar - isoscalars of even parity) made hermitian using a prescription for constructing hermitian (anti-)commutators discussed in [5], consisting of $q\ \chi_-$'s, $P[\ ,\ ]$'s, j (which can take only the values 0 or 1) $\epsilon^{\mu\nu\rho\lambda}$'s, $kv^{(s)}_{\mu\nu}$'s, $tF^+_{\rho\lambda}$'s and $uF^-_{\mu\nu}$'s, for which the following phase rule is satisfied, are the only terms allowed:

$$(-1)^{q+P+j+k+t+u} = 1. \tag{6}$$

In [5] for $k = t = u = 0$, (6) was used to generate complete lists up to $O(q^3)$ in the absence of external vector and axial-vector fields. In this paper, the same phase rule is used to construct complete lists of $O(q^4)$ including external fields.

Let A, B, C, D be operators chosen from the pion-field dependent building blocks of (4). In what follows, and especially in Section 3, use will be made of a notation of [11]: $(A, B) \equiv [A, B]$ or $[A, B]_+$. One can then show that apart from the (0,0,0,2,0,0,0)-, (0,0,2,0,0,0,0)-, (0,0,0,0,2,0,0)-, (0,0,0,0,0,0,2)- and (0,0,0,0,1,0,1)-type terms (using the notation of (4)), the following is the complete list of $O(q^4, \phi^{2n})$ terms (using (6)):

$$
\begin{aligned}
(i) \quad & (A,(B,(C,D))) \equiv (a)[A,[B,[C,D]_+]];\ (b)[A,[B,[C,D]]_+];\\
& (c)[A,[B,[C,D]]]_+;\ (d)[A,[B,[C,D]_+]_+]_+;\\
(ii) \quad & ((A,B),(C,D)) \equiv (a)[[A,B],[C,D]_+];\ (b)[[A,B]_+,[C,D]];\\
& (c)[[A,B],[C,D]]_+;\ (d)[[A,B]_+,[C,D]_+]_+;\\
(iii) \quad & i(A,(B,(C,D))) \equiv (a)i[A,[B,[C,D]_+]_+];\ (b)i[A,[B,[C,D]_+]]_+;\\
& (c)i[A,[B,[C,D]]_+]_+;\ (d)i[A,[B,[C,D]]];\\
(iv) \quad & i((A,B),(C,D)) \equiv (a)i[[A,B],[C,D]];\ (b)i[[A,B]_+,[C,D]_+];\\
& (c)i[[A,B],[C,D]_+]_+;\ (d)i[[A,B]_+,[C,D]]_+;\\
(v) \quad & i(A,(B,C)) \equiv (a)i[A,[B,C]]_+;\ (b)i[A,[B,C]_+];\ (c)A \leftrightarrow B;\\
& (d)i[[A,B],C]_+;\ (e)i[[A,B]_+,C];\\
(vi) \quad & (A,(B,C)) \equiv (a)[A,[B,C]];\ (b)[A,[B,C]_+]_+;\ (c)A \leftrightarrow B\\
& (d)[[A,B],C];\ (e)[[A,B]_+,C]_+,
\end{aligned} \tag{7}
$$

where it is understood that of all the possible terms implied by $(i)(A,(B,(C,D)))$, $(i)((A,B),(C,D))$ and $(i)(A,(B,C))$, only those that are allowed by (6) are to be included. For (0,0,0,2,0,0,0)-, (0,0,2,0,0,0,0)-, (0,0,0,0,2,0,0)-, (0,0,0,0,0,0,2)- and (0,0,0,0,1,0,1)-type terms, one needs to include [1]

$$
\begin{aligned}
& (A,A) \equiv (\chi_+,\chi_+);\ (\chi_-,\chi_-);\ (F^+_{\mu\nu},F^+_{\rho\lambda});\ (v^{(s)}_{\mu\nu},v^{(s)}_{\rho\lambda});\\
& (A,B) \equiv [F^+_{\mu\nu},v^{(s)}_{\rho\lambda}]_+.
\end{aligned} \tag{8}
$$

The list (7) holds good for $O(q^4, \phi^{2n+1})$ terms with the difference that there is an additional factor of i multiplying the terms in (i), (ii) and (vi), and the i in (iii), (iv) and (v), is absent. The reason for including i only in some combinations of terms has to do with imposing charge conjugation invariance along with other symmetries *directly within HBChPT* (See [5]). The terms of (7) and their analogs for $O(q^4, \phi^{2n+1})$ are not all independent since they can be related by a number of linear relations: see next section (and [5] for $O(q^3)$).

3 Further Reduction due to Algebraic identities

In this section, we discuss further reduction in addition to the ones obtained from (6). The main result from [5] is that one need not consider trace-dependent terms in SU(2) HBChPT if

[1] One need not consider (0,0,0,0,0,2,0). See (10) and the discussion thereafter.

one assumes isospin conservation. Given that isospin violation enters only via $\chi_\pm$, thus, $\chi_\pm$-independent trace-dependent $\mathrm{O}(q^4)$ terms can be eliminated in preference for $\chi_\pm$-independent trace-independent terms. We discuss reduction due to algebraic identities in the various categories of (7). Some of the algebraic reductions require one to consider more than one category at a time, e.g., for $\mathrm{O}(q^4, \phi^{2n})$ terms, the generalized Jacobi identities in (15) require one to consider $(i), (i)(A \leftrightarrow B), (ii)$.

One can show that ([11]):

$$[\mathrm{D}_\mu, \mathrm{D}_\nu] = \frac{1}{4}[u_\mu, u_\nu] - \frac{i}{2}F^+_{\mu\nu} - iv^{(s)}_{\mu\nu}, \tag{9}$$

$$[\mathrm{D}_\mu, u_\nu] - [\mathrm{D}_\nu, u_\mu] = F^-_{\mu\nu}. \tag{10}$$

The first relation, referred to as the curvature relation, will be used extensively in conjunction with some generalized Jacobi identities and Schouten's identity discussed below. As a consequence of (9), we will choose to always write $[\mathrm{D}_\mu, \mathrm{D}_\nu]$ in terms of $[u_\mu, u_\nu], F^+_{\mu\nu}, v^{(s)}_{\mu\nu}$. As a consequence of (10), we see that $F^-_{\mu\nu}$ can be eliminated in preference for $[\mathrm{D}_{[\mu}, u_{\nu]}]$. However, the relative coefficients of $[u_\mu, u_\nu], F^+_{\mu\nu}, v^{(s)}_{\mu\nu}$, as well as the relative coefficients of $[\mathrm{D}_\mu, u_\nu], [\mathrm{D}_\nu, u_\mu]$ can be made arbitrary because each can be obtained from the nonrelativistic reduction of linearly independent terms. One interesting consequence of (10) is that in the absence of external fields, the commutator of the covariant derivative and the axial-vector building block is symmetric in the Lorentz indeces - something missed in [1].

Further, another source of major reduction in number of terms is the Schouten's identity:

$$\epsilon^{\mu_1\mu_2\mu_3\mu_4} X^{\mu_5} + \text{cyclic} = 0, \tag{11}$$

where X_μ is any arbitrary (axial)vector.

It is because of the curvature relation that one requires to consider, e.g., some (4,0,0,0,0,0,0)-, (0,4,0,0,0,0,0)- (2,2,0,0,0,0,0)-, (2,0,0,0,1,0,0)-, (2,0,0,0,0,0,1)-, (0,2,0,0,1,0,0)-, (0,2,0,0,0,0,1)-, (0,0,0,0,2,0,0)-, (0,0,0,0,0,0,2)- and (0,0,0,0,1,0,1)-type terms together in (18). Due to Schouten's identity, e.g., the aforementioned 7-tuples with $\epsilon^{\mu\nu\rho\lambda} v_\rho \mathrm{S}_\lambda$ and $\epsilon^{\mu\nu\rho\lambda} \mathrm{S}_\lambda v\cdot$ (where v_ρ is contracted with a building block) are required to be considered together.

Finally, the $\mathrm{O}(q^2)$ pion eom will be used for reduction in the number of linearly independent terms:

$$[\mathrm{D}_\mu, u^\mu] = \frac{i}{2}(\chi_- - \frac{1}{2}\langle\chi_-\rangle) \equiv \frac{i}{2}\tilde{\chi}_-. \tag{12}$$

3.1 $\mathrm{O}(q^4, \phi^{2n})$ Terms

In this subsection, we consider reduction in the number of independent $\mathrm{O}(q^4, \phi^{2n})$ terms due to various algebraic identities. The following are the algebraic identities responsible for reduction in number of $\mathrm{O}(q^4, \phi^{2n})$ terms: (13), (14), (15), (20), (16) (23) and (17). For (15) and (20), there are two sets each of terms (one $\epsilon^{\mu\nu\rho\lambda}$-dependent and the other $\epsilon^{\mu\nu\rho\lambda}$-independent), that need to be considered.

$\underline{\underline{p = q = 0 \text{ in (4): } (A, (B, (C, D))); ((A, B), (C, D))}}$

This includes $(i)-(iv)$ of (7). All terms in each of the first four types (of terms) in (7) [$(i)-(iv)$] are linearly independent for unequal field operators A, B, C, D. However for (4,0,0,0,0,0,0), (0,4,0,0,0,0,0) and (2,2,0,0,0,0,0), L.C.-independent terms, one needs to consider A=C, B=D in (i) in equation (7). Using

$$[A,[B,[A,B]_+]] = -[A,[B,[A,B]]_+] \tag{13}$$

only three of the four terms in (i) of equation (7), are linearly independent. Similarly, using

$$[[A,B],[A,B]_+] = -[[A,B]_+,[A,B]], \tag{14}$$

only three of the four terms in (ii) of equation (7), are linearly independent.

There are some reductions possible due to some generalized Jacobi identities by considering : $(i),(i)(A\leftrightarrow B),(ii)$ of (7)($\equiv \epsilon^{\mu\nu\rho\lambda}$ -independent terms), and $(iii),(iii)(A\leftrightarrow B),(iv)$ of (7) ($\equiv \epsilon^{\mu\nu\rho\lambda}$-dependent terms). The reason why one can not hope to get reductions by considering any other pairs of types of terms in $(i)-(iv)$ (in (7)), is because one can get (linear) algebraic relationships only between those terms which are (both) independent of (have) an overall factor of i.

$\underline{(i),(i)(A\leftrightarrow B),(ii) \text{ of } (7)}$

One can show the following 6 generalized Jacobi identities:

$$\begin{aligned}
&[A,[B,[C,D]_+]] - [[A,B],[C,D]_+] = (i)(a)(A\leftrightarrow B)\\
&[A,[B,[C,D]_+]] - [[A,B]_+,[C,D]_+]_+ = -(i)(d)(A\leftrightarrow B)\\
&[A,[B,[C,D]]_+] - [[A,B]_+,[C,D]] = -(i)(b)(A\leftrightarrow B)\\
&[A,[B,[C,D]]_+] - [[A,B],[C,D]]_+ = (i)(c)(A\leftrightarrow B)\\
&[A,[B,[C,D]]]_+ - [[A,B]_+,[C,D]] = -(i)(c)(A\leftrightarrow B)\\
&[A,[B,[C,D]_+]_+]_+ - [[A,B],[C,D]_+] = (i)(d)(A\leftrightarrow B).
\end{aligned} \tag{15}$$

Further, one can apply the following to $(B,(C,D))$ contained in $(A,(B,(C,D)))$:

$$\begin{aligned}
&[B,[C,D]] - [[B,C],D] = [C,[B,D]]\\
&[B,[C,D]_+]_+ - [[B,C]_+,D]_+ = -[C,[B,D]]\\
&[B,[C,D]] - [[B,C]_+,D]_+ = -[C,[B,D]_+]_+,
\end{aligned} \tag{16}$$

and:

$$\begin{aligned}
&i[B,[C,D]]_+ - i[[B,C],D]_+ = i[C,[B,D]_+]\\
&i[B,[C,D]]_+ - i[[B,C]_+,D] = -i[C,[B,D]]_+\\
&i[B,[C,D]_+] - i[[B,C]_+,D] = -i[C,[B,D]_+].
\end{aligned} \tag{17}$$

The three identities in (17) are similar to the ones that occur in SUSY graded Lie algebra for $B,D\equiv$ fermionic and $C\equiv$ bosonic fields, $B,C\equiv$ fermionic and $D\equiv$ bosonic fields, and $B,C,D\equiv$ fermionic fields, respectively.

(1) Using (15), (9), (16) and (17), one needs to consider the following (4,0,0,0,0,0,0)-, (0,4,0,0,0,0,0)- (2,2,0,0,0,0,0)-, (2,0,0,0,1,0,0)-, (2,0,0,0,0,0,1)-, (0,2,0,0,1,0,0)-, (0,2,0,0,0,0,1)-, (0,0,0,0,2,0,0)-, (0,0,0,0,0,0,2)- and (0,0,0,0,1,0,1)-type $\epsilon^{\mu\nu\rho\lambda}$-independent terms together:

$$
\begin{array}{l}
(v\cdot\mathrm{D},(\mathrm{D}_\mu,(v\cdot\mathrm{D},\mathrm{D}^\mu))),\ (\mathrm{D}_\mu,(v\cdot\mathrm{D},(v\cdot\mathrm{D},\mathrm{D}^\mu))),\ ((v\cdot\mathrm{D},\mathrm{D}_\mu),(v\cdot\mathrm{D},\mathrm{D}^\mu))\\
(v\cdot u,(u_\mu,(v\cdot u,u^\mu))),\ (u_\mu,(v\cdot u,(v\cdot u,u^\mu))),\ ((v\cdot u,u_\mu),(v\cdot u,u^\mu)),\\
(v\cdot\mathrm{D},(\mathrm{D}_\mu,(v\cdot u,u^\mu))),\ (\mathrm{D}_\mu,(v\cdot\mathrm{D},(u_\mu,v\cdot u))),\ ((v\cdot\mathrm{D},\mathrm{D}_\mu),(v\cdot u,u^\mu))),\\
(v\cdot u,(u_\mu,(v\cdot\mathrm{D},\mathrm{D}^\mu))),\ (u_\mu,(v\cdot u,(v\cdot\mathrm{D},\mathrm{D}^\mu))),\\
(v\cdot\mathrm{D},((\mathrm{D}_\mu,v\cdot u),u^\mu)),\ (u^\mu,((\mathrm{D}_\mu,v\cdot u),v\cdot\mathrm{D}),\ ((v\cdot\mathrm{D},u^\mu),(\mathrm{D}_\mu,v\cdot u)),\\
(\mathrm{D}_\mu,((v\cdot\mathrm{D},u^\mu),v\cdot u)),\ (v\cdot u,((v\cdot\mathrm{D},u^\mu),\mathrm{D}_\mu)),\\
v^\nu(v\cdot\mathrm{D},(\mathrm{D}_\mu,F^{+\ \mu\nu})),\ v^\nu(\mathrm{D}_\mu,(v\cdot\mathrm{D},F^{+\ \mu\nu})),v^\nu((v\cdot\mathrm{D},\mathrm{D}_\mu),F^{+\ \mu\nu}),\\
v^\nu(v\cdot\mathrm{D},(\mathrm{D}_\mu,v^{(s)\ \mu\nu})),\ v^\nu(\mathrm{D}_\mu,(v\cdot\mathrm{D},v^{(s)\ \mu\nu})),v^\nu((v\cdot\mathrm{D},\mathrm{D}_\mu),v^{(s)\ \mu\nu}),\\
v^\nu(v\cdot u,(u_\mu,F^{+\ \mu\nu})),\ v^\nu(u_\mu,(v\cdot u,F^{+\ \mu\nu})),v^\nu((v\cdot u,u_\mu),F^{\mu\nu}),\\
v^\nu(v\cdot u,(u_\mu,v^{(s)\ \mu\nu})),\ v^\nu(u_\mu,(v\cdot u,v^{(s)\ \mu\nu})),v^\nu((v\cdot u,u_\mu),v^{(s)\ \mu\nu}),\\
v^\kappa F^+_{\kappa\mu}v_\sigma F^{+\ \sigma\mu},\ v^\kappa v^{(s)}_{\kappa\mu}v_\sigma F^{+\ \sigma\mu},\ v^\kappa v^{(s)}_{\kappa\mu}v_\sigma v^{(s)\ \sigma\mu}.
\end{array}
\tag{18}
$$

One needs to do a careful counting of the total number of identities that one can write down using (15) and (9), and the total number of terms in those identities.

(2) Similarly, one will need to consider (4,0,0,0,0,0,0)-, (0,4,0,0,0,0,0)- (2,2,0,0,0,0,0)-, (2,0,0,0,1,0,0)- , (2,0,0,0,0,0,1)-, (0,2,0,0,1,0,0)-, (0,2,0,0,0,0,1)-, (0,0,0,0,2,0,0)-, (0,0,0,0,0,0,2)- and (0,0,0,0,1,0,1)-type terms together:

$$
\begin{array}{l}
(\mathrm{D}_\nu,(\mathrm{D}_\mu,(\mathrm{D}^\nu,\mathrm{D}^\mu))),\ ((\mathrm{D}_\nu,\mathrm{D}_\mu),(\mathrm{D}^\nu,\mathrm{D}^\mu))\\
(u_\nu,(u_\mu,(u^\nu,u^\mu))),\ ((u_\nu,u_\mu),(u^\nu,u^\mu)),\\
(\mathrm{D}_\nu,(\mathrm{D}_\mu,(u^\nu,u^\mu))),\ ((\mathrm{D}_\nu,\mathrm{D}_\mu),(u^\nu,u^\mu)))\\
(u_\nu,(u_\mu,(\mathrm{D}^\nu,\mathrm{D}^\mu)))\\
(\mathrm{D}^\mu,(\mathrm{D}^\nu,F^+_{\mu\nu})),\ ([\mathrm{D}^\mu,\mathrm{D}^\nu],F^+_{\mu\nu})),\\
(\mathrm{D}^\mu,(\mathrm{D}^\nu,v^{(s)}_{\mu\nu})),\ ([\mathrm{D}^\mu,\mathrm{D}^\nu],v^{(s)}_{\mu\nu})),\\
(u^\mu,(u^\nu,F^+_{\mu\nu})),\ ([u^\mu,u^\nu],F^+_{\mu\nu})),\\
(u^\mu,(u^\nu,v^{(s)}_{\mu\nu})),\ ([u^\mu,u^\nu],v^{(s)}_{\mu\nu}))\\
(F^+_{\mu\nu})^2,\ v^{(s)}_{\mu\nu}F^{+\ \mu\nu},\ (v^{(s)}_{\mu\nu})^2
\end{array}
\tag{19}
$$

$(iii), (iii)(A\leftrightarrow B)$ and (iv); (vi) of (7)

One can show the following generalized Jacobi identities to be true:

$$
\begin{array}{l}
i[A,[B,[C,D]_+]_+]-i[[A,B]_+,[C,D]_+]=-(iii)(a)(A\leftrightarrow B)\\
i[A,[B,[C,D]_+]_+]-i[[A,B],[C,D]_+]_+=(iii)(b)(A\leftrightarrow B)\\
i[A,[B,[C,D]_+]]_+-i[[A,B]_+,[C,D]_+]=-(iii)(b)(A\leftrightarrow B)\\
i[A,[B,[C,D]]_+]_+-i[[A,B],[C,D]]=(iii)(c)(A\leftrightarrow B)
\end{array}
$$

$$
\begin{aligned}
&i[A,[B,[C,D]]_+]_+ - i[[A,B]_+,[C,D]_+] = -(iii)(d)(A\leftrightarrow B)\\
&i[A,[B,[C,D]]] - i[[A,B],[C,D]] = (iii)(d)(A\leftrightarrow B)
\end{aligned}
\tag{20}
$$

Again one can apply the generalized Jacobi-iieuntities (16) and (17) to $(B,(C,D))$ contained in $(A,(B,(C,D)))$.

(3) The identities (20), (16) and (17) along with (9) require one to consider the following category of $\epsilon^{\mu\nu\rho\lambda}$-dependent (4,0,0,0,0,0,0)-, (0,4,0,0,0,0,0)- (2,2,0,0,0,0,0)-, (2,0,0,0,1,0,0)-, (2,0,0,0,0,0,1)-, (0,2,0,0,1,0,0)-, (0,2,0,0,0,0,1)-, (0,0,0,0,2,0,0)-, (0,0,0,0,0,0,2)- and (0,0,0,0,1,0,1)-type terms together:

$$
\begin{aligned}
&i\epsilon^{\mu\nu\rho\lambda}v_\rho\Big[(\mathrm{D}_\mu,(\mathrm{D}_\nu,(\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}))),\ (u_\mu,(u_\nu,(u_\lambda,\mathrm{S}\cdot u))),\\
&(\mathrm{S}\cdot\mathrm{D},(\mathrm{D}_\mu,(\mathrm{D}_\nu,[\mathrm{D}_\nu,\mathrm{D}_\lambda]))\ (\mathrm{D}_\mu,(\mathrm{S}\cdot\mathrm{D},[\mathrm{D}_\nu,\mathrm{D}_\lambda])),\\
&(\mathrm{S}\cdot u,(u_\mu,([u_\nu,u_\lambda])),\ (u_\mu,(\mathrm{S}\cdot u,[u_\nu,u_\lambda])),\\
&([u_\mu,u_\nu],(\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D})),\ (u_\mu,(u_\nu,(\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D}))),\\
&(\mathrm{D}_\mu,(\mathrm{S}\cdot\mathrm{D},[u_\nu,u_\lambda])),\ (\mathrm{S}\cdot\mathrm{D},(\mathrm{D}_\mu,[u_\nu,u_\lambda])),\\
&(u_\mu,(\mathrm{S}\cdot u,[\mathrm{D}_\nu,\mathrm{D}_\lambda])),\ (\mathrm{S}\cdot u,(u_\mu,[\mathrm{D}_\nu,\mathrm{D}_\lambda])),\\
&(\mathrm{D}_\mu,(\mathrm{D}_\nu,(u_\lambda,\mathrm{S}\cdot u))),\ ([\mathrm{D}_\mu,\mathrm{D}_\nu],(u_\lambda,\mathrm{S}\cdot u))\\
&(\mathrm{D}_\mu,((\mathrm{D}_\nu,u_\lambda),\mathrm{S}\cdot u)),\ (\mathrm{S}\cdot u,((\mathrm{D}_\nu,u_\lambda),\mathrm{D}_\mu));\\
&((\mathrm{D}_\mu,\mathrm{S}\cdot u),(\mathrm{D}_\nu,u_\lambda));\\
&(\mathrm{D}_\mu,((\mathrm{D}_\nu,\mathrm{S}\cdot u),u_\lambda)),\ (u_\lambda,((\mathrm{D}_\nu,\mathrm{S}\cdot u),\mathrm{D}_\mu)),\\
&i(F^+_{\mu\nu},(\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D})),\ i(\mathrm{D}_\lambda,(\mathrm{S}\cdot\mathrm{D},F^+_{\mu\nu})),\ i(\mathrm{S}\cdot\mathrm{D},(\mathrm{D}_\lambda,F^+_{\mu\nu})),\\
&i(v^{(s)}_{\mu\nu},(\mathrm{D}_\lambda,\mathrm{S}\cdot\mathrm{D})),\ i(\mathrm{D}_\lambda,(\mathrm{S}\cdot\mathrm{D},v^{(s)}_{\mu\nu})),\ i(\mathrm{S}\cdot\mathrm{D},(\mathrm{D}_\lambda,v^{(s)}_{\mu\nu})),\\
&i(\mathrm{D}_\mu,(\mathrm{D}_\nu,F^+_{\kappa\lambda}))S^\kappa,\ i([\mathrm{D}_\mu,\mathrm{D}_\nu],F^+_{\kappa\lambda})S^\kappa,\ i(\mathrm{D}_\mu,(\mathrm{D}_\nu,v^{(s)}_{\kappa\lambda}))S^\kappa,\\
&i(u_\mu,(u_\nu,F^+_{\kappa\lambda}))S^\kappa,\ i([u_\mu,u_\nu],F^+_{\kappa\lambda})S^\kappa,\ i(u_\mu,(u_\nu,v^{(s)}_{\kappa\lambda}))S^\kappa,\\
&i(F^+_{\mu\nu},(u_\lambda,\mathrm{S}\cdot u)),\ i(u_\lambda,(\mathrm{S}\cdot u,F^+_{\mu\nu})),\ i(\mathrm{S}\cdot u,(u_\lambda,F^+_{\mu\nu})),\\
&i[u_\lambda,\mathrm{S}\cdot u]_+v^{(s)}_{\mu\nu},\ [F^+_{\mu\nu},F^+_{\kappa\lambda}]S^\kappa\Big].
\end{aligned}
\tag{21}
$$

Analogous to (18) and (19), one needs to do a careful counting of the total number of identities that one can write down using (20), and (9), and the total number of terms in those identities.

A similar analysis can be carried out for terms with $v\leftrightarrow\mathrm{S}$ in (21). However, due to (11), this set of terms has to be considered in conjunction with (22) $(u_\kappa,\mathrm{D}^\kappa\to v\cdot u, v\cdot\mathrm{D})$

The identities in (16) are also used in, e.g., $\epsilon^{\mu\nu\rho\lambda}$-dependent (1,1,0,1,0,0,0)-type terms.

(4) The identities (20), (16), (17) along with the curvature relation, require one to consider the following category of $\epsilon^{\mu\nu\rho\lambda}$- dependent (4,0,0,0,0,0,0)-, (0,4,0,0,0,0,0)- (2,2,0,0,0,0,0)-, (2,0,0,0,1,0,0)-, (2,0,0,0,0,0,1)-, (0,2,0,0,1,0,0)-, (0,2,0,0,0,0,1)-, (0,0,0,0,2,0,0)-, (0,0,0,0,0,0,2)-

type terms together:

$$
\begin{aligned}
& i\epsilon^{\mu\nu\rho\lambda} v_\rho \mathrm{S}_\lambda \Big((u_\kappa, (u_\mu, (u^\kappa, u_\nu))),\ (u_\mu, (u_\kappa, (u^\kappa, u_\nu))),\ ((u_\mu, u_\kappa), (u_\nu, u^\kappa)), \\
& (\mathrm{D}_\kappa, (\mathrm{D}_\mu, (\mathrm{D}^\kappa, \mathrm{D}_\nu))),\ ((\mathrm{D}_\mu, (\mathrm{D}_\kappa, (\mathrm{D}^\kappa, \mathrm{D}_\nu))),\ ((\mathrm{D}_\kappa, \mathrm{D}_\mu), (\mathrm{D}^\kappa, \mathrm{D}_\nu)), \\
& ((\mathrm{D}_\kappa, (\mathrm{D}_\mu, (u^\kappa, u_\nu))),\ ((\mathrm{D}_\mu, (\mathrm{D}_\kappa, (u^\kappa, u_\nu))),\ ((\mathrm{D}_\kappa, \mathrm{D}_\mu), (u^\kappa, u_\nu)), \\
& (u_\mu, (u_\kappa, (\mathrm{D}^\kappa, \mathrm{D}_\nu))),\ (u_\mu, (u_\kappa, (\mathrm{D}^\kappa, \mathrm{D}_\nu))), \\
& (\mathrm{D}_\mu, ((\mathrm{D}_\nu, u_\kappa), u^\kappa)),\ ((\mathrm{D}_\mu, u^\kappa), (\mathrm{D}_\nu, u^\kappa)), \\
& (u^\kappa, ((\mathrm{D}_\mu, u^\kappa), \mathrm{D}_\nu)), \\
& i(\mathrm{D}^\kappa, (\mathrm{D}_\mu, F^+_{\kappa\nu})),\ i(\mathrm{D}_\mu, (\mathrm{D}^\kappa, F^+_{\kappa\nu})),\ i((\mathrm{D}^\kappa, \mathrm{D}_\mu), F^+_{\kappa\nu}) \\
& i(\mathrm{D}^\kappa, (\mathrm{D}_\mu, v^{(s)}_{\kappa\nu})),\ i(\mathrm{D}_\mu, (\mathrm{D}^\kappa, v^{(s)}_{\kappa\nu})),\ i((\mathrm{D}^\kappa, \mathrm{D}_\mu), v^{(s)}_{\kappa\nu}) \\
& i(u^\kappa, (u_\mu, F^+_{\kappa\nu})),\ i(u_\mu, (u^\kappa, F^+_{\kappa\nu})),\ i((u^\kappa, u_\mu), F^+_{\kappa\nu}) \\
& i(u^\kappa, (u_\mu, v^{(s)}_{\kappa\nu})),\ i(u_\mu, (u^\kappa, v^{(s)}_{\kappa\nu})),\ i((u^\kappa, u_\mu), v^{(s)}_{\kappa\nu}) \\
& i[F^{+\ \kappa}_{\mu}, F^+_{\kappa\nu}].\Big).
\end{aligned}
\tag{22}
$$

A similar analysis can be carried out for terms with $(u_\kappa, \mathrm{D}^\kappa) \to (v\cdot u, v\cdot \mathrm{D})$ in (22).

For (0,4,0,0,0,0,0)- and (1,3,0,0,0,0,0)-type terms, writing $u_\mu = u^a_\mu \tau^a$, $[\mathrm{D}_\mu, u_\nu] = [\mathrm{D}_\mu, u_\nu]^a \tau^a$ (where $[\mathrm{D}_\mu, u_\nu]^a \equiv \partial_\mu u^a_\nu + i\epsilon^{abc}\Gamma^b_\mu u^c_\nu$), one will need to consider the following relations:

$$
\begin{aligned}
(a)\quad & [\tau^b, [\tau^c, \tau^d]_+] = [\tau^a, [\tau^b, [\tau^c, \tau^d]]_+] = 0; \\
& [[\tau^a, \tau^b], [\tau^c, \tau^d]_+] = 0; \\
& i[[\tau^a, \tau^b]_+, [\tau^c, \tau^d]_+] = 0; \\
(b)\quad & [u_\mu, u_\nu]^2_+ - [u_\mu, u_\nu]^2 = 4u^4; \\
(c)\quad & v\cdot u u_\mu v\cdot u u^\mu + h.c. = -2u^2(v\cdot u)^2 + [v\cdot u, u_\mu]^2_+; \\
(d)\quad & i\epsilon^{\mu\nu\rho\lambda}[[u_\mu, u_\kappa], u_\nu, u^\kappa]_+]_+ = i\epsilon^{\mu\nu\rho\lambda}\Big(u^2[u_\mu, u_\nu] - u^\kappa[u_\mu, u_\nu]u_\kappa\Big); \\
(e)\quad & i\epsilon^{\mu\nu\rho\lambda}[[u_\mu, v\cdot u], [u_\nu, v\cdot u]_+]_+ = i\epsilon^{\mu\nu\rho\lambda}\Big((v\cdot u)^2[u_\mu, u_\nu] - v\cdot[u_\mu, u_\nu]v\cdot u\Big).
\end{aligned}
\tag{23}
$$

At least one of k, p, q, t, u is $\neq 0$ in (4):$(A, (B, C))$

This includes (v) and (vi) of (7).

(v) of (7)

The identities in (17) are used in, e.g., $\epsilon^{\mu\nu\rho\lambda}$-independent (1,1,0,1,0,0,0)-type terms. When applying (17) to (2,0,1,0,0,0,0), because of (9), one will need to consider the following terms together:

$$
\begin{aligned}
& i\epsilon^{\mu\nu\rho\lambda} v_\rho \mathrm{S}_\lambda \Big((\mathrm{D}_\mu, (\mathrm{D}_\nu, \chi_+)),\ ([\mathrm{D}_\mu, \mathrm{D}_\nu], \chi_+); \\
& (u_\mu, (u_\nu, \chi_+)),\ ([u_\mu, u_\nu], \chi_+),\ i(F^+_{\mu\nu}, \chi_+),\ iv^{(s)}_{\mu\nu}, \chi_+)\Big).
\end{aligned}
\tag{24}
$$

3.2 $\mathbf{O}(q^4, \phi^{2n+1})$ Terms

In this subsection, we consider the reduction in the number of $\mathrm{O}(q^4, \phi^{2n+1})$ terms because of algebraic identities. The discussion in this subsection will be much briefer than the preceding (subsection).

$(i) - (iv)$ of $(7)'$

(1) The identities (15) are the same for $\mathrm{O}(q^4, \phi^{2n+1})$ except for an overall factor of i. We will denote the analogue of (15) for $\mathrm{O}(q^4, \phi^{2n+1})$ terms as $(15)'$.[2] Using it together with and (9), one sees that one needs to consider the following set of terms together:

$$
\begin{aligned}
(a)\quad & i(v\cdot \mathrm{D}, (\mathrm{D}_\mu, (\mathrm{D}^\mu, \mathrm{S}\cdot u))),\ i(\mathrm{D}_\mu, (v\cdot \mathrm{D}, (\mathrm{D}^\mu, \mathrm{S}\cdot u))),\ i((v\cdot \mathrm{D}, \mathrm{D}_\mu), (\mathrm{D}^\mu, \mathrm{S}\cdot u)),\\
& i(\mathrm{S}\cdot u, (\mathrm{D}_\mu, (\mathrm{D}^\mu, v\cdot \mathrm{D}))),\ i(\mathrm{D}_\mu, (\mathrm{S}\cdot u, (\mathrm{D}^\mu, v\cdot \mathrm{D})));\\
& (\mathrm{D}^\mu, (\mathrm{S}\cdot u, F^+_{\mu\nu}))v^\nu, (\mathrm{S}\cdot u, (\mathrm{D}^\mu, F^+_{\mu\nu}))v^\nu, ((\mathrm{D}^\mu, \mathrm{S}\cdot u), F^+_{\mu\nu})v^\nu,\\
& (\mathrm{D}^\mu, (\mathrm{S}\cdot u, v^{(s)}_{\mu\nu}))v^\nu, (\mathrm{S}\cdot u, (\mathrm{D}^\mu, v^{(s)}_{\mu\nu}))v^\nu, ((\mathrm{D}^\mu, \mathrm{S}\cdot u), v^{(s)}_{\mu\nu})v^\nu;\\
& i(v\cdot u, (u_\mu, (\mathrm{S}\cdot u, \mathrm{D}^\mu))),\ i(u_\mu, (v\cdot u, (\mathrm{S}\cdot u, \mathrm{D}^\mu))),\ i((v\cdot u, u_\mu), (\mathrm{S}\cdot u, \mathrm{D}^\mu)),\\
& i(\mathrm{D}_\mu, (\mathrm{S}\cdot u, (u^\mu, v\cdot u)))\ i(\mathrm{S}\cdot u, (\mathrm{D}_\mu, (u^\mu, v\cdot u)));\\
& i(\mathrm{D}_\mu, (\mathrm{D}^\mu, (v\cdot \mathrm{D}, \mathrm{S}\cdot u))),\ i(\mathrm{D}^2, (v\cdot \mathrm{D}, \mathrm{S}\cdot u)),\\
& i(\mathrm{S}\cdot u, (v\cdot \mathrm{D}, \mathrm{D}^2)),\ i(v\cdot \mathrm{D}, (\mathrm{S}\cdot u, \mathrm{D}^2)).\\
(b)\quad & i(v\cdot \mathrm{D}, (\mathrm{D}_\mu, (\mathrm{S}\cdot \mathrm{D}, u^\mu))),\ i(\mathrm{D}_\mu, (v\cdot \mathrm{D}, (\mathrm{S}\cdot \mathrm{D}, u^\mu))),\ i((v\cdot \mathrm{D}, \mathrm{D}_\mu), (\mathrm{S}\cdot \mathrm{D}, u^\mu)),\\
& i(u_\mu, (\mathrm{S}\cdot \mathrm{D}, (\mathrm{D}^\mu, v\cdot \mathrm{D})))\ i(\mathrm{S}\cdot \mathrm{D}, (u_\mu, (\mathrm{D}^\mu, v\cdot \mathrm{D})));\\
& (u^\mu, (\mathrm{S}\cdot \mathrm{D}, F^+_{\mu\nu}))v^\nu, (\mathrm{S}\cdot \mathrm{D}, (u^\mu, F^+_{\mu\nu}))v^\nu, ((u^\mu, \mathrm{S}\cdot \mathrm{D}), F^+_{\mu\nu})v^\nu,\\
& (u^\mu, (\mathrm{S}\cdot \mathrm{D}, v^{(s)}_{\mu\nu}))v^\nu, (\mathrm{S}\cdot \mathrm{D}, (u^\mu, v^{(s)}_{\mu\nu}))v^\nu, (u^\mu, \mathrm{S}\cdot \mathrm{D}), v^{(s)}_{\mu\nu})v^\nu\\
& i(v\cdot u, (u_\mu, (u^\mu, \mathrm{S}\cdot \mathrm{D}))),\ i(u_\mu, (v\cdot u, (u^\mu, \mathrm{S}\cdot \mathrm{D}))),\ i((v\cdot u, u_\mu), (u^\mu, \mathrm{S}\cdot \mathrm{D})),\\
& i(\mathrm{S}\cdot \mathrm{D}, (u_\mu, (u^\mu, v\cdot u))),\ i(u_\mu, (\mathrm{S}\cdot \mathrm{D}, (u^\mu, v\cdot u)))\\
& i(u^\mu, (u_\mu, (v\cdot u, \mathrm{D}^\mu))),\ i(\mathrm{S}\cdot u, (u_\mu, v\cdot u, \mathrm{D}^\mu)),\\
& i((u_\mu, \mathrm{S}\cdot u), (v\cdot u, \mathrm{D}^\mu)),\ i(\mathrm{D}_\mu, (v\cdot u, (u^\mu, \mathrm{S}\cdot u))),\\
& i(v\cdot u, (\mathrm{D}_\mu, (u^\mu, \mathrm{S}\cdot u)));\\
& i(u^\mu, (u_\mu, (v\cdot u, \mathrm{S}\cdot \mathrm{D}))),\ i(u^2, (v\cdot u, \mathrm{S}\cdot \mathrm{D})),\\
& i(\mathrm{S}\cdot \mathrm{D}, (v\cdot u, u^2)),\ i(v\cdot u, (\mathrm{S}\cdot \mathrm{D}, u^2));\\
(c)\quad & i(v\cdot \mathrm{D}, (\mathrm{S}\cdot \mathrm{D}, (\mathrm{D}^\mu, u_\mu))),\ i(\mathrm{S}\cdot \mathrm{D}, (v\cdot \mathrm{D}, (\mathrm{D}^\mu, u_\mu))),\ i((v\cdot \mathrm{D}, \mathrm{S}\cdot \mathrm{D}), (\mathrm{D}^\mu, u_\mu)),\\
& i(u_\mu, (\mathrm{D}^\mu, (v\cdot \mathrm{D}, \mathrm{S}\cdot \mathrm{D}))),\ i(\mathrm{D}_\mu, (u^\mu, (v\cdot \mathrm{D}\mathrm{S}\cdot \mathrm{D})));\\
& v^{[\rho}\mathrm{S}^{\lambda]}(\mathrm{D}_\mu, (u^\mu, F^+_{\rho\lambda})), v^{[\rho}\mathrm{S}^{\lambda]}(u^\mu, (\mathrm{D}_\mu, F^+_{\rho\lambda})), v^{[\rho}\mathrm{S}^{\lambda]}((\mathrm{D}_\mu, u^\mu), F^+_{\rho\lambda})\\
& v^{[\rho}\mathrm{S}^{\lambda]}(\mathrm{D}_\mu, (u^\mu, v^{(s)}_{\rho\lambda})), v^{[\rho}\mathrm{S}^{\lambda]}(u^\mu, (\mathrm{D}_\mu, v^{(s)}_{\rho\lambda})), v^{[\rho}\mathrm{S}^{\lambda]}((\mathrm{D}_\mu, u^\mu), v^{(s)}_{\rho\lambda})\\
& i(v\cdot u, (\mathrm{S}\cdot u, (u^\mu, \mathrm{D}_\mu))),\ i(\mathrm{S}\cdot u, (v\cdot u, (\mathrm{D}^\mu, u_\mu))),\ i((v\cdot u, \mathrm{S}\cdot u), (\mathrm{D}^\mu, u_\mu)),\\
& i(\mathrm{D}_\mu, (u^\mu, (v\cdot u, \mathrm{S}\cdot u))),\ i(u_\mu, (\mathrm{D}^\mu, (v\cdot u, \mathrm{S}\cdot u)));\\
(d)\quad & i(\mathrm{D}_\mu, (\mathrm{S}\cdot \mathrm{D}, (v\cdot \mathrm{D}, u^\mu))),\ i(\mathrm{S}\cdot \mathrm{D}, (\mathrm{D}_\mu, (v\cdot \mathrm{D}, u^\mu))),\ i((\mathrm{S}\cdot \mathrm{D}, \mathrm{D}_\mu), (v\cdot \mathrm{D}, u^\mu)),\\
& i(u_\mu, (v\cdot \mathrm{D}, (\mathrm{D}^\mu, \mathrm{S}\cdot \mathrm{D}))),\ i(v\cdot \mathrm{D}, (u_\mu, (\mathrm{D}^\mu, \mathrm{S}\cdot \mathrm{D}))),
\end{aligned}
$$

[2]Similarly, the analogs of (7), (20), (17) and (16) will be denoted by $(7)'$, $(20)'$, $(17)'$ and $(16)'$.

$$
\begin{aligned}
& (u^\mu, (v\cdot \mathrm{D}, F^+_{\mu\nu}))\mathrm{S}^\nu, (v\cdot \mathrm{D}, (u^\mu, F^+_{\mu\nu}))\mathrm{S}^\nu,\ ((u^\mu, v\cdot \mathrm{D}), F^+_{\mu\nu})\mathrm{S}^\nu, \\
& (u^\mu, (v\cdot \mathrm{D}, v^{(s)}_{\mu\nu}))\mathrm{S}^\nu, (v\cdot \mathrm{D}, (u^\mu, v^{(s)}_{\mu\nu}))\mathrm{S}^\nu,\ ((u^\mu, v\cdot \mathrm{D}), v^{(s)}_{\mu\nu})\mathrm{S}^\nu, \\
& i(u_\mu, (\mathrm{S}\cdot u, (u^\mu, v\cdot \mathrm{D}))),\ i(\mathrm{S}\cdot u, (u_\mu, (u^\mu, v\cdot \mathrm{D}))),\ i((u_\mu, \mathrm{S}\cdot u), (u^\mu, v\cdot \mathrm{D}))), \\
& i(v\cdot \mathrm{D}, (u_\mu, (u^\mu, \mathrm{S}\cdot u)),\ i(u_\mu, (v\cdot \mathrm{D}, (u^\mu, \mathrm{S}\cdot u)); \\
& i(u_\mu, (u^\mu, (\mathrm{S}\cdot u, v\cdot \mathrm{D}))),\ i(u^2, (\mathrm{S}\cdot u, v\cdot \mathrm{D})), \\
& i(v\cdot \mathrm{D}, (\mathrm{S}\cdot u, u^2)),\ i(\mathrm{S}\cdot u, (v\cdot \mathrm{D}, u^2)); \\
(e)\quad & i(\mathrm{D}_\mu, (\mathrm{S}\cdot \mathrm{D}, (\mathrm{D}^\mu, v\cdot u))),\ i(\mathrm{S}\cdot \mathrm{D}, (\mathrm{D}_\mu, (\mathrm{D}^\mu, v\cdot u))),\ i((\mathrm{D}_\mu, \mathrm{S}\cdot \mathrm{D}), (\mathrm{D}^\mu, v\cdot u))), \\
& i(v\cdot u, (\mathrm{D}_\mu, (\mathrm{D}^\mu, \mathrm{S}\cdot \mathrm{D})),\ i(\mathrm{D}_\mu, (v\cdot u, (\mathrm{D}^\mu, \mathrm{S}\cdot \mathrm{D})); \\
& (v\cdot u, (\mathrm{D}^\mu, F^+_{\mu\nu}))\mathrm{S}^\nu, (\mathrm{D}^\mu, (v\cdot u, F^+_{\mu\nu}))\mathrm{S}^\nu,\ ((v\cdot u, \mathrm{D}^\mu), F^+_{\mu\nu})\mathrm{S}^\nu, \\
& (v\cdot u, (\mathrm{D}^\mu, v^{(s)}_{\mu\nu}))\mathrm{S}^\nu, (\mathrm{D}^\mu, (v\cdot u, v^{(s)}_{\mu\nu}))\mathrm{S}^\nu,\ ((v\cdot u, \mathrm{D}^\mu), v^{(s)}_{\mu\nu})\mathrm{S}^\nu, \\
& i(u_\mu, (\mathrm{S}\cdot u, (v\cdot u, \mathrm{D}^\mu))),\ i(\mathrm{S}\cdot u, (u_\mu, (v\cdot u, \mathrm{D}^\mu))),\ i((\mathrm{S}\cdot u, u_\mu), (v\cdot u, \mathrm{D}^\mu)), \\
& i(\mathrm{D}_\mu, (v\cdot u, (u^\mu, \mathrm{S}\cdot u))),\ i(v\cdot u, (\mathrm{D}_\mu, (u^\mu, \mathrm{S}\cdot u))); \\
& i(\mathrm{D}_\mu, (\mathrm{D}^\mu, (\mathrm{S}\cdot \mathrm{D}, v\cdot u))),\ i(\mathrm{D}^2, (\mathrm{S}\cdot \mathrm{D}, v\cdot u)), \\
& i(v\cdot u, (\mathrm{S}\cdot \mathrm{D}, \mathrm{D}^2)),\ i(\mathrm{S}\cdot \mathrm{D}, (v\cdot u, \mathrm{D}^2)).
\end{aligned}
\tag{25}
$$

Using $(16)'$ and $(17)'$, one needs to consider $(a)-(e)$ together.

Similarly, using $(15)'$ and $(9)'$, one needs to consider simultaneously

$$
\begin{aligned}
(a)\quad & i(v\cdot \mathrm{D}, (\mathrm{S}\cdot \mathrm{D}, (v\cdot \mathrm{D}, v\cdot u))),\ i(\mathrm{S}\cdot \mathrm{D}, (v\cdot \mathrm{D}, (v\cdot \mathrm{D}, v\cdot u))),\ i((v\cdot \mathrm{D}, \mathrm{S}\cdot \mathrm{D}), (v\cdot \mathrm{D}, v\cdot u))), \\
& i(v\cdot u, (v\cdot \mathrm{D}, (v\cdot \mathrm{D}, \mathrm{S}\cdot \mathrm{D})),\ i(v\cdot \mathrm{D}, (v\cdot u, (v\cdot \mathrm{D}, \mathrm{S}\cdot \mathrm{D})); \\
& v^{[\rho}\mathrm{S}^{\lambda]}(v\cdot \mathrm{D}, (v\cdot u, F^+_{\rho\lambda})), v^{[\rho}\mathrm{S}^{\lambda]}(v\cdot u, (v\cdot \mathrm{D}, F^+_{\rho\lambda})), v^{[\rho}\mathrm{S}^{\lambda]}((v\cdot \mathrm{D}, v\cdot u), F^+_{\rho\lambda}); \\
& v^{[\rho}\mathrm{S}^{\lambda]}(v\cdot \mathrm{D}, (v\cdot u, v^{(s)}_{\rho\lambda})), v^{[\rho}\mathrm{S}^{\lambda]}(v\cdot u, (v\cdot \mathrm{D}, v^{(s)}_{\rho\lambda})), v^{[\rho}\mathrm{S}^{\lambda]}((v\cdot \mathrm{D}, v\cdot u), v^{(s)}_{\rho\lambda}); \\
& i(v\cdot u, (\mathrm{S}\cdot u, (v\cdot u, v\cdot \mathrm{D}))),\ i(\mathrm{S}\cdot u, (v\cdot u, (v\cdot u, v\cdot \mathrm{D}))),\ i((\mathrm{S}\cdot u, v\cdot u), (v\cdot u, v\cdot \mathrm{D})), \\
& i(v\cdot u, (v\cdot \mathrm{D}, (v\cdot u, \mathrm{S}\cdot u)),\ i(v\cdot \mathrm{D}, (v\cdot u, (v\cdot u, \mathrm{S}\cdot u)); \\
& i(v\cdot u, (\mathrm{S}\cdot \mathrm{D}, (v\cdot \mathrm{D})^2)),\ i(\mathrm{S}\cdot \mathrm{D}, (v\cdot u, (v\cdot \mathrm{D})^2)), \\
& i((v\cdot \mathrm{D})^2, (\mathrm{S}\cdot \mathrm{D}, v\cdot u)),\ i(v\cdot \mathrm{D}, (v\cdot \mathrm{D}, (\mathrm{S}\cdot \mathrm{D}, v\cdot u))); \\
& i(v\cdot \mathrm{D}, \mathrm{S}\cdot u, (v\cdot u)^2)),\ i(\mathrm{S}\cdot u, (v\cdot \mathrm{D}, (v\cdot u)^2)), \\
& i((v\cdot u)^2, (\mathrm{S}\cdot u, v\cdot \mathrm{D})),\ i(v\cdot u, (v\cdot u, (\mathrm{S}\cdot u, v\cdot \mathrm{D})); \\
(b)\quad & i(v\cdot \mathrm{D}, (v\cdot \mathrm{D}, (v\cdot \mathrm{D}, \mathrm{S}\cdot u))),\ i((v\cdot \mathrm{D})^2, (v\cdot \mathrm{D}, \mathrm{S}\cdot u)), \\
& i[\mathrm{S}\cdot u, (v\cdot \mathrm{D})^3]_+,\ i(v\cdot \mathrm{D}, (\mathrm{S}\cdot u, (v\cdot \mathrm{D})^2)).
\end{aligned}
\tag{26}
$$

Using $(16)'$ and $(17)'$, one needs to consider (a) and (b) together.

(2) Using $(20)'$, $(16)'$, $(17)'$, (9) and (11), one sees that one has to consider the following sets of terms together:

$$
\begin{aligned}
(a)\quad & \epsilon^{\mu\nu\rho\lambda}\Big((\mathrm{D}_\mu, (\mathrm{D}_\nu, (\mathrm{D}_\rho, u_\lambda))),\ ([\mathrm{D}_\mu, \mathrm{D}_\nu], (\mathrm{D}_\rho, u_\lambda))), \\
& (u_\mu, (\mathrm{D}_\nu, [\mathrm{D}_\rho, \mathrm{D}_\lambda])),\ (\mathrm{D}_\mu, (u_\nu, [\mathrm{D}_\rho, \mathrm{D}_\lambda])); \\
& i(\mathrm{D}_\mu, (u_\nu, F^+_{\rho\lambda})), i(u_\nu, (\mathrm{D}_\mu, F^+_{\rho\lambda})), i((u_\nu, \mathrm{D}_\mu), F^+_{\rho\lambda});
\end{aligned}
$$

$$
\begin{aligned}
&i(u_\mu,(\mathrm{D},v^{(s)}_{\rho\lambda})),i(\mathrm{D}_\mu,u_\nu)v^{(s)}_{\rho\lambda};\\
&(u_\mu,(u_\nu,(u_\rho,\mathrm{D}_\lambda))),\ ([u_\mu,u_\nu],(u_\rho,\mathrm{D}_\lambda)),\\
&(\mathrm{D}_\mu,(u_\nu,[u_\rho,u_\lambda])),\ (u_\mu,(\mathrm{D}_\nu,[u_\rho,u_\lambda]))\Big);
\end{aligned}
$$

$$
\begin{aligned}
(b)\quad &\epsilon^{\mu\nu\rho\lambda}v_\rho\Big((v\cdot\mathrm{D},(\mathrm{D}_\mu,(\mathrm{D}_\nu,u_\lambda))),\ ((v\cdot\mathrm{D},\mathrm{D}_\mu),(\mathrm{D}_\nu,u_\lambda))),\ (\mathrm{D}_\mu,(v\cdot\mathrm{D},(\mathrm{D}_\nu,u_\lambda))),\\
&(u_\mu,(\mathrm{D}_\nu,(v\cdot\mathrm{D},\mathrm{D}_\lambda))),\ (\mathrm{D}_\mu,(u_\nu,(v\cdot\mathrm{D},\mathrm{D}_\lambda)));\\
&(v\cdot u,(u_\mu,(u_\nu,\mathrm{D}_\lambda))),\ (u_\mu,(v\cdot u,(u_\nu,\mathrm{D}_\lambda))),\ ((u_\mu,v\cdot u),(u_\nu,\mathrm{D}_\lambda)),\\
&(\mathrm{D}_\mu,(u_\nu,(v\cdot u,u_\lambda))),\ (u_\mu,(\mathrm{D}_\nu,(v\cdot u,u_\lambda)))\Big),\\
&i(\mathrm{D}_\mu,(u_\nu,F^+_{\kappa\lambda}))v^\kappa,i(u_\nu,(\mathrm{D}_\mu,F^+_{\kappa\lambda}))v^\kappa,i((u_\nu,\mathrm{D}_\mu),F^+_{\kappa\lambda})v^\kappa;\\
&i(u_\mu,(\mathrm{D},v^{(s)}_{\kappa\lambda}))v^\kappa,i(\mathrm{D}_\mu,u_\nu)v^{(s)}_{\kappa\lambda}v^\kappa;\\
(c)\quad &\epsilon^{\mu\nu\rho\lambda}v_\rho\Big((\mathrm{D}_\mu,(\mathrm{D}_\nu,(v\cdot\mathrm{D},u_\lambda))),([\mathrm{D}_\mu,\mathrm{D}_\nu],(v\cdot\mathrm{D},u_\lambda));\\
&(v\cdot\mathrm{D},(u_\lambda,[\mathrm{D}_\mu,\mathrm{D}_\nu])),(u_\mu,(v\cdot\mathrm{D},[\mathrm{D}_\mu,\mathrm{D}_\nu]));\\
&(u_\mu,(u_\nu,(v\cdot\mathrm{D},u_\lambda))),([u_\mu,u_\nu],(v\cdot\mathrm{D},u_\lambda)));\\
&(u_\lambda,(v\cdot\mathrm{D},[u_\mu,u_\nu])),(v\cdot\mathrm{D},(u_\lambda,[u_\mu,u_\nu]))\\
&(v\cdot\mathrm{D},(u_\lambda,F^+_{\mu\nu})),(u_\lambda,(v\cdot\mathrm{D},F^+_{\mu\nu})),((u_\lambda,v\cdot\mathrm{D}),F^+_{\mu\nu}));\\
&(v\cdot\mathrm{D},(u_\lambda,v^{(s)}_{\mu\nu})),(u_\lambda,(v\cdot\mathrm{D},v^{(s)}_{\mu\nu})),((u_\lambda,v\cdot\mathrm{D}),v^{(s)}_{\mu\nu}));\\
(d)\quad &\epsilon^{\mu\nu\rho\lambda}v_\rho\Big((\mathrm{D}_\mu,(\mathrm{D}_\nu,(\mathrm{D}_\lambda,v\cdot u))),([\mathrm{D}_\mu,\mathrm{D}_\nu],(v\cdot u,\mathrm{D}_\lambda));\\
&(\mathrm{D}_\lambda,(v\cdot u,[\mathrm{D}_\mu,\mathrm{D}_\nu])),(v\cdot u,(\mathrm{D}_\lambda,[\mathrm{D}_\mu,\mathrm{D}_\nu]));\\
&(u_\mu,(u_\nu,(v\cdot u,\mathrm{D}_\lambda))),([u_\mu,u_\nu],(v\cdot u,\mathrm{D}_\lambda)));\\
&(v\cdot u,(\mathrm{D}_\lambda,[u_\mu,u_\nu])),(\mathrm{D}_\lambda,(v\cdot u,[u_\mu,u_\nu]));\\
&(\mathrm{D}_\lambda,(v\cdot u,F^+_{\mu\nu})),(v\cdot u,(\mathrm{D}_\lambda,F^+_{\mu\nu})),((v\cdot u,\mathrm{D}_\lambda),F^+_{\mu\nu}));\\
&(\mathrm{D}_\lambda,(v\cdot u,v^{(s)}_{\mu\nu})),(v\cdot u,(\mathrm{D}_\lambda,v^{(s)}_{\mu\nu})),((v\cdot u,\mathrm{D}_\lambda),v^{(s)}_{\mu\nu})).
\end{aligned}
\tag{27}
$$

At least one of k,p,q,t,u is $\neq 0$ in (4):$(A,(B,C))$

(v) of $(7)'$

Using (9) and $(17)'$, one sees that one will have to consider the following set of terms together:

$$
\begin{aligned}
&(\mathrm{S}\cdot\mathrm{D},(v\cdot\mathrm{D},\chi_-)),\ (v\cdot\mathrm{D},(\mathrm{S}\cdot\mathrm{D},\chi_-)),\ ((v\cdot\mathrm{D},\mathrm{S}\cdot\mathrm{D}),\chi_-);\\
&i[F^+_{\mu\nu},\chi_-]_+v^{[\mu}\mathrm{S}^{\nu]},iv^{(s)}_{\mu\nu}\chi_-v^{[\mu}\mathrm{S}^{\nu]};\\
&(v\cdot u,(\mathrm{S}\cdot u,\chi_-)),\ (\mathrm{S}\cdot u,(v\cdot u,\chi_-)),\ ((\mathrm{S}\cdot u,v\cdot u),\chi_-).
\end{aligned}
\tag{28}
$$

(vi) of $(7)'$

Using $(16)'$, one sees that one will have to consider the following set of terms together:

$$
i(\mathrm{S}\cdot\mathrm{D},(v\cdot u,\chi_+)),\ i(v\cdot u,(\mathrm{S}\cdot\mathrm{D},\chi_+)),\ i((\mathrm{S}\cdot\mathrm{D},v\cdot u),\chi_+);v\leftrightarrow\mathrm{S}.
\tag{29}
$$

Note that because of parity constraints and the algebra of the S_μs (See [5]), there are no Levi Civita-dependent (2,0,0,1,0,0,0)-, (0,2,0,1,0,0,0)- and (1,1,1,0,0,0,0)-type terms.

3.3 Isospin Violation

As noted in Section 3, isospin violation enters via $\chi_\pm$, we thus need to reconsider the (m, n, p, q, t, u, k) with p or $q \neq 0$. We will also have to include trace-dependent terms for these type of terms. The following identities are used in arriving at terms in Table 2:

$$
\begin{aligned}
&u^\mu\chi_+u_\mu + u^2\chi_+ + \text{h.c.} = 2u^2\langle\chi_+\rangle + u^\mu\langle[u_\mu,\chi_+]_+\rangle,\\
&v\cdot u\chi_+v\cdot u + u^2\chi_+ + \text{h.c.} = 2(v\cdot u)^2\langle\chi_+\rangle + v\cdot u\langle[v\cdot u,\chi_+]_+\rangle,\\
&[\chi_\pm,\chi_\pm]_+ = 2\langle\chi_\pm\rangle\chi_\pm + \frac{1}{2}\langle[\chi_\pm,\chi_\pm]_+\rangle - \frac{1}{2}\langle\chi_\pm\rangle^2,\\
&[F^+_{\mu\nu},\chi_\pm]_+ = \frac{1}{2}[F^+_{\mu\nu},\langle\chi_\pm\rangle]_+ + \frac{1}{2}\langle[F^+_{\mu\nu},\chi_\pm]_+\rangle,\\
&i[v\cdot \mathrm{D},[\mathrm{S}\cdot u,\chi_+]_+]_+ = \frac{i}{2}[v\cdot \mathrm{D},[\mathrm{S}\cdot u,\langle\chi_+\rangle]_+]_+ + \frac{i}{2}[v\cdot \mathrm{D},\langle[\mathrm{S}\cdot u,\chi_+]_+\rangle,\\
&i[v\cdot \mathrm{D},[\mathrm{S}\cdot u,\chi_+]_+]_+ = \frac{i}{2}[v\cdot \mathrm{D},[\mathrm{S}\cdot u,\langle\chi_+\rangle]_+]_+ + \frac{i}{2}[v\cdot \mathrm{D},\langle[\mathrm{S}\cdot u,\chi_+]_+\rangle,\\
&+\frac{i}{2}[v\cdot \mathrm{D},\langle[\mathrm{S}\cdot u,\chi_+]_+\rangle,\\
&\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([\mathrm{D}_\mu,[u_\nu,\chi_-]_+]_+ = [\mathrm{D}_\mu,[u_\nu,\langle\chi_-\rangle]_+]_+ + \frac{1}{2}[\mathrm{D}_\mu,\langle[u_\nu,\chi_-]_+\rangle]_+\Big),\\
&i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda\Big([[u_\mu,u_\nu],\chi_+]_+ = [[u_\mu,u_\nu],\langle\chi_+\rangle]_+ + \frac{1}{2}[[u_\mu,u_\nu],^H\chi_+]_+\rangle\Big),\\
&i[\mathrm{D}_\mu,[u^\mu,\chi_-]_+] = \frac{i}{2}\langle[\mathrm{D}_\mu,[u^\mu,\chi_-]_+]\rangle + \frac{i}{2}[\mathrm{D}_\mu,[u^\mu,\langle\chi_-\rangle]_+],\\
&i[[\mathrm{D}_\mu,u^\mu],\chi_-]_+ = \frac{i}{2}\langle[[\mathrm{D}_\mu,u^\mu],\chi_-]_+\rangle + \frac{i}{2}[[\mathrm{D}_\mu,u^\mu],\langle\chi_-\rangle]_+,\\
&i[u_\mu,[\mathrm{D}^\mu,\chi_-]]_+ = \frac{i}{2}\langle[u_\mu,[\mathrm{D}^\mu,\chi_-]]_+\rangle + \frac{i}{2}[u_\mu,[\mathrm{D}^\mu,\langle\chi_-\rangle]_+.
\end{aligned}
\tag{30}
$$

4 The Lists of Independent Terms in $\mathcal{L}_{\mathrm{HBChPT}}$ (off-shell nucleons)

In this section, using (6), and the algebraic reductions of Section 3, we list all possible $\mathcal{A}$-type terms of $\mathrm{O}(q^4,\phi^{2n})$, and $\mathrm{O}(q^4,\phi^{2n+1})$ in Tables 1 and 2, that are allowed by (6) and have not been eliminated in Section 3. As noted in Section 2 (and [5]), for off-shell nucleons, $\gamma^0\mathcal{B}^\dagger\gamma^0\mathcal{C}^{-1}\mathcal{B}\in\mathcal{A}$. Hence, it is sufficient to list only $\mathcal{A}$-type terms (for off-shell nucleons).

Using the algebraic identities of Section 3, if we end up with m independent identities in $n(>m)$ terms, then we can take $(n-m)$ linearly independent terms. Even though the phase rule (6) and linear independence of terms are sufficient for listing terms in the $\mathrm{O}(q^4)$ HBChPT Lagrangian for off-shell nucleons, however, if for a given choice of terms and group of terms in (7), we find similar group of terms in [8], then while listing the $(n-m)$ terms, preference is given to including terms that also figure in Table 1 of [8]. The reason for doing the same is that this allows for an easy identification of the finite terms, given that the divergent (counter) terms have been worked out in [8].

In tables 1 (that one gets if one assumes isospin symmetry) and 2 (that one gets if one includes isospin violation), the allowed 7-tuples (m, n, p, q, t, u, k) are listed along with the corresponding terms. The main aim is to find the number of finite $\mathrm{O}(q^4)$ terms, given that the UV divergent terms have already been worked out in [8]. For this purpose, the terms in tables 1 and 2 are labeled as F denoting the finite terms and D denoting the divergent terms. For the purpose of comparison with [8], we have also indicated which terms in table 1 of [8] the D-type terms correspond to. The LECs of $\mathrm{O}(q^4)$ terms in [8] are denoted by $d_i, i = 1$ to 199. Further, the $i = 188$ term in Table 1 of [8] should have S_ρ instead of v_ρ.

Overall, one gets 27 finite and 79 divergent (counter) terms at $\mathrm{O}(q^4)$.

5 On-shell reduction

In this section, we discuss the derivation of the on-shell $\mathrm{O}(q^4)$ $\mathcal{L}_{\mathrm{HBChPT}}$, directly within HBChPT using the techniques of [5].

The main result obtained in [5] extended to include external fields in the context of complete on-shell reduction within HBChPT was the following rule:

$$\begin{aligned}
&\mathcal{A}-\text{type terms of the form } \bar{\mathrm{H}}\mathrm{S}\cdot\mathrm{D}\mathcal{O}\mathrm{H}\ +\ \text{h.c.}\\
&\text{or } \bar{\mathrm{H}}\mathrm{v}\cdot\mathrm{D}\mathcal{O}\mathrm{H}\ +\ \text{h.c.}\\
&\text{or } \bar{\mathrm{H}}\mathcal{O}^\mu\mathrm{D}_\mu\mathrm{H}\ +\ \text{h.c. can be eliminated}\\
&\text{except for } \mathcal{O}_\mu \equiv \left(i^{m_1+l_5+l_7+t+u+1},\ \text{or } \epsilon^{\nu\lambda\kappa\rho}\times\Omega\right)\times u_\mu\Lambda\\
&\text{with } l_1 \geq 1, \Omega \equiv 1(i) \text{ for } (-)^{m_1+l_5+l_7+t+u+1} = -1(1),\\
&\text{or}\\
&\mathcal{O}_\mu \equiv \left(i^{m_1+l_5+l_7+k+t+u},\ \text{or } \epsilon^{\nu\lambda\kappa\rho}\times\Omega'\right)\times \mathrm{D}_\mu\Lambda,\\
&\text{with } l_1 \geq 1, \Omega' \equiv 1(i) \text{ for } (-)^{m_1+l_5+l_7+k+t+u} = -1(1).
\end{aligned}\tag{31}$$

In (31)

$$\Lambda \equiv \prod_{i=1}^{M_1}\mathcal{V}_{\nu_i}\prod_{j=1}^{M_2}u_{\rho_j}(v\cdot u)^{l_1}u^{2l_2}\chi_+^{l_3}\chi_-^{l_4}([v\cdot\mathrm{D},\ ])^{l_5}(\mathrm{D}_\beta\mathrm{D}^\beta)^{l_6}(u_\alpha\mathrm{D}^\alpha)^{l_7}(v^{(s)}_{\sigma\omega})^k(F^+_{\rho\lambda})^t(F^+_{\mu\nu})^u\mathrm{S}^r_\kappa,\tag{32}$$

where $\mathcal{V}_{\nu_i} \equiv v_{\nu_i}$ or $\mathrm{D}_{\nu_\mathrm{i}}$. where $\mathcal{V}_{\nu_i} \equiv v_{\nu_i}$ or $\mathrm{D}_{\nu_\mathrm{i}}$. The number of D_{ν_i}s in (32) equals $m_1(\leq M_1)$. Assuming that Lorentz invariance, parity and hermiticity have been implemented, the choice of the factors of i in (31) automatically incorporates the phase rule (6). In (31), it is only the contractions of the building blocks that has been indicated. Also, traces have not been indicated. It is understood that all (anti-)commutators in the HBChPT Lagrangian are to be expanded out until one hits the first D_μ, so that the $\mathcal{A}$-type HBChPT term can be put in the form $\bar{\mathrm{H}}\mathcal{O}^\mu\mathrm{D}_\mu\mathrm{H}$ +h.c.

The complete on-shell O(q^4) HBChPT Lagrangian can be shown to be given by:

$$\mathcal{L}^{(4)}_{\mathrm{HBChPT}} = A^{(4)} + \frac{1}{2\mathrm{m}}\gamma^0 B^{(2)\ \dagger}\gamma^0 B^{(2)}$$
$$+\frac{1}{2\mathrm{m}}\Big[\gamma^0 B^{(3)\ \dagger}\gamma^0 B^{(1)} + \gamma^0 B^{(1)\ \dagger}\gamma^0 B^{(3)}\Big]$$
$$-\frac{1}{4\mathrm{m}^2}\Big[\gamma^0 B^{(2)\ \dagger}\gamma^0 C^{(1)} B^{(1)} + \gamma^0 B^{(1)\ \dagger}\gamma^0 C^{(1)} B^{(2)}\Big]$$
$$-\frac{1}{4\mathrm{m}^2}\gamma^0 B^{(1)\ \dagger}\gamma^0 C^{(2)} B^{(1)} + \frac{1}{8\mathrm{m}^3}\gamma^0 B^{(1)\ \dagger}\gamma^0 (C^{(1)})^2 B^{(1)}. \tag{33}$$

Using

$$B^{(2)}_{\mathrm{OS}}[v^{(s)}, F^+, \chi_-] = \alpha_3\gamma^5\chi_3 i\alpha_7\gamma^5 v^{[\mu}\mathrm{S}^{\nu]}v^{(s)}_{\mu\nu} + i\alpha_8\gamma^5 v^{[\mu}\mathrm{S}^{\nu]}F^+_{\mu\nu} + \alpha_9\rangle\chi_-\rangle, \tag{34}$$

(OS≡on-shell) one gets:

$$\frac{1}{2\mathrm{m}}\gamma^0 B^{(2)\ \dagger}\gamma^0 B^{(2)}[v^{(s)}, F^+, \chi_-]$$
$$= \frac{1}{2\mathrm{m}}\Big[+(\alpha_3\chi_- + \alpha_9\langle\chi_-\rangle)^2 - \alpha_1[[v\cdot u, \mathrm{S}\cdot u], (\alpha_- + \alpha_9\langle\chi_-\rangle)]_+$$
$$-v^{[\mu}\mathrm{S}^{\nu]}[(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu}, (\alpha_3\chi_- + \alpha_9\langle\chi_-\rangle)]_+ + i\alpha_4[(\alpha_3\chi_- + \alpha_9\langle\chi_-\rangle), [v\cdot \mathrm{D}, v\cdot u]]_+$$
$$+(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu}(\alpha_7 v^{(s)\nu}_{\rho} + \alpha_8 F^{+\nu}_{\rho}) - \frac{i}{2}\epsilon^{\nu\lambda\alpha\beta}v_\alpha v^\mu v^\rho \mathrm{S}_\beta(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})(\alpha_7 v^{(s)}_{\rho\lambda} + \alpha_8 F^+_{\rho\lambda})$$
$$i\alpha_1\alpha_7\Big(-\frac{1}{4}v^\mu[[v\cdot u, u^\nu], v^{(s)}_{\mu\nu}]_+\Big) + i\alpha_1\alpha_8\Big(-\frac{1}{4}v^\mu[[v\cdot u, u^\nu], F^+_{\mu\nu}]_+ + \frac{1}{2}\epsilon^{\nu\rho\alpha\beta}v_\alpha \mathrm{S}_\beta v^\mu[[v\cdot u, u_\rho], F^+_{\mu\nu}]\Big)$$
$$+i\alpha_2\alpha_7\Big(-\frac{1}{2}\epsilon^{\nu\rho\alpha\beta}v_\alpha \mathrm{S}_\beta v^\mu[[v\cdot u, u^\rho]_+, v^{(s)}_{\mu\nu}]_+\Big)$$
$$+i\alpha_2\alpha_8\Big(\frac{1}{4}v^\mu[[v\cdot u, u^\nu]_+, F^+_{\mu\nu}] - \frac{1}{2}\epsilon^{\nu\rho\alpha\beta}v_\alpha \mathrm{S}_\beta v^\mu[[v\cdot u, u^\rho]_+, F^+_{\mu\nu}]_+\Big)$$
$$-\alpha_3 v^{[\mu}\mathrm{S}^{\nu]}[(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu}), \chi_-]_+ - \alpha_4 v^{[\mu}\mathrm{S}^{\nu]}[(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu}), [v\cdot \mathrm{D}, v\cdot u]]_+$$
$$-i\alpha_2\alpha_4[[v\cdot u, \mathrm{S}\cdot u]_+, [v\cdot \mathrm{D}, v\cdot u]] + i\alpha_3\alpha_4[\chi_-, [v\cdot \mathrm{D}, v\cdot u]]_+\Big]. \tag{35}$$

Using (34) and $C^{(1)}_{\mathrm{OS}}$, and eliminating all terms proportional to the nonrelativistic eom by field redefinition of H, one gets:

$$-\frac{1}{4\mathrm{m}^2}\gamma^0 B^{(2)\ \dagger}\gamma^0 C^{(1)} B^{(1)}[v^{(s)}, F^+] + \mathrm{h.c.}$$
$$= (\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})\Big[-2iv^\mu[v\cdot \mathrm{D}, \mathrm{D}^\nu] + g^0_A \mathrm{D}^\nu \mathrm{S}\cdot u + 2\epsilon^{\nu\rho\lambda\sigma}v^\mu v_\lambda \mathrm{S}_\sigma[v\cdot \mathrm{D}, \mathrm{D}_\rho]$$
$$-\frac{i}{2}g^0_A\epsilon^{\nu\rho\lambda\sigma}v^\mu v_\lambda \mathrm{D}_\rho u_\sigma - g^0_A v^\mu \mathrm{S}^\nu[v\cdot \mathrm{D}, v\cdot u] - \frac{g^{0\ 2}_A}{4}v\cdot u u^\nu v^\mu$$
$$+\frac{ig^{0\ 2}_A}{2}\epsilon^{\alpha\nu\rho\omega}v^\mu v_\rho \mathrm{S}_\omega v\cdot u u_\alpha + g^0_A \mathrm{S}^\nu v^\mu \mathrm{D}\cdot u - g^0_A v^\mu \mathrm{S}\cdot \mathrm{D} u^\nu$$

$$-g_A^0 v^\mu \mathrm{S}^\nu [v\cdot \mathrm{D}, v\cdot u] + \frac{i}{2}\epsilon^{\rho\lambda\alpha\nu} v^\mu v_\alpha u_\rho \mathrm{D}_\lambda + g_A^0 v^\mu u^\nu \mathrm{S}\cdot\mathrm{D} - g_A^0 v^\mu \mathrm{S}\cdot u \mathrm{D}^\nu$$
$$+\frac{i{g_A^0}^2}{2}(-\frac{1}{2}v^\mu u^\nu + i\epsilon^{\nu\rho\lambda\sigma}v^\mu v_\lambda \mathrm{S}_\sigma u_\rho)\Big]$$
$$+\Big(2(\alpha_3\chi_- + \alpha_9\langle\chi_-\rangle)v\cdot \mathrm{DS}\cdot\mathrm{D} - i\frac{g_A^0}{2}(\alpha_3\chi_- + \alpha_9\langle\chi_-\rangle)[v\cdot\mathrm{D}, v\cdot u] + \frac{{g_A^0}^2}{2}(\alpha_3\chi_- + \alpha_9\langle\chi_-\rangle)v\cdot u\mathrm{S}\cdot u$$
$$+g_A^0(\alpha_3\chi_- + \alpha_9\langle\chi_-\rangle)\Big[\frac{1}{4}v\cdot u\mathrm{S}\cdot u + \frac{i}{4}u_\mu \mathrm{D}^\mu + \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda u_\mu \mathrm{D}_\nu\Big] + \mathrm{h.c.}\Big] \tag{36}$$

Similarly, using:

$$C_{\mathrm{S}}^{(2)}[v^{(s)}, F^+, \chi_+] = -\alpha_6\chi_+\alpha_7\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda v^{(s)}_{\mu\nu} + \alpha_8\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda F^+_{\mu\nu} - \alpha_{10}\langle\chi_+\rangle \tag{37}$$

and

$$B^{(1)} = -2i\gamma^5 \mathrm{S}\cdot\mathrm{D} - \frac{g_A^0}{2}\gamma^5 v\cdot u, \tag{38}$$

and eliminating all terms proportional to the nonrelativistic eom by field redefinition of H, one sees that:

$$-\frac{1}{4\mathrm{m}^2}\gamma^0 B^{(1)\ \dagger}\gamma^0 C^{(2)} B^{(1)}[v^{(s)}, F^+, \chi_+]$$
$$= -\frac{1}{4\mathrm{m}^2}\Big[-\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}\cdot\mathrm{D}(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})\mathrm{D}_\lambda + \Big(i\mathrm{D}^\mu(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})\mathrm{D}^\nu$$
$$+g_A^0 \mathrm{S}\cdot u(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})v^\mu \mathrm{D}^\nu + g_A^0 v^\mu \mathrm{D}^\nu(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})\mathrm{S}\cdot u\Big)$$
$$-\frac{{g_A^0}^2}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho v\cdot u(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})\mathrm{S}_\lambda v\cdot u$$
$$+2\epsilon^{\mu\nu\rho\lambda}v_\rho\Big(-\frac{ig_A^0}{4}v^\omega \mathrm{D}_\mu(\alpha_7 v^{(s)}_{\omega\nu} + \alpha_8 F^+_{\omega\nu})u_\lambda - \mathrm{D}_\mu(\alpha_7 v^{(s)}_{\omega\nu} + \alpha_8 F^+_{\omega\nu})\mathrm{D}^\omega \mathrm{S}_\lambda\Big)$$
$$-g_A^0\Big(\mathrm{S}\cdot\mathrm{D}(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})u^\nu v^\sigma - \mathrm{D}^\rho v^\sigma(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})u_\rho$$
$$+\frac{ig_A^0}{4}u^\nu v^\sigma(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})v\cdot u - \frac{g_A^0}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda u_\mu(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})v^\sigma v\cdot u$$
$$+\Big(\frac{ig_A^0}{4}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{D}_\lambda(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})v\cdot u - \frac{i{g_A^0}^2}{4}v^\mu u^\nu(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})v\cdot u$$
$$-\frac{{g_A^0}^2}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda v^\kappa u_\mu(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})v\cdot u$$
$$-g_A^0 \mathrm{D}^\nu(\alpha_7 v^{(s)}_{\mu\nu} + \alpha_8 F^+_{\mu\nu})\mathrm{S}^\nu + h.c.\Big)$$
$$+\Big({g_A^0}^2\Big[\frac{1}{4}v\cdot u(\alpha_6\chi_+ + \alpha_{10}\langle\chi_+\rangle)v\cdot u - \frac{1}{4}u_\mu(\alpha_6\chi_+ + \alpha_{10}\langle\chi_+\rangle)u^\mu + \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda u_\mu(\alpha_6\chi_+ + \alpha_{10}\langle\chi_+\rangle)u_\nu\Big]$$
$$+\mathrm{D}_\mu\chi_+\mathrm{D}^\mu - 2i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda \mathrm{D}_\mu\chi_+\mathrm{D}_\nu$$
$$+(ig_A^0 v\cdot u\chi_+ \mathrm{S}\cdot\mathrm{D} + \mathrm{h.c.}) + \frac{{g_A^0}^2}{4}v\cdot u\chi_+ v\cdot u\Big)\Big] \tag{39}$$

Using:

$$
\begin{aligned}
&B^{(3)}_{\mathrm{OS}}[v^{(s)}, F^+, \chi_+, \chi_-] = \beta_4\gamma^5[v\cdot u, \chi_+]_+ + \beta_{14}\gamma^5[\chi_-, \mathrm{S}\cdot u]\\
&+\beta_{17}\gamma^5\mathrm{S}^\nu[\mathrm{D}^\mu, v^{(s)}_{\mu\nu}] i\beta_{18}\gamma^5 v^\mu[F^+_{\mu\nu}, u^\nu] + \beta_{19}\epsilon^{\mu\nu\rho\lambda}\gamma^5\mathrm{S}_\lambda[F^+_{\mu\nu}, u_\rho]_+\\
&+\beta_{20}\epsilon^{\mu\nu\rho\lambda}\gamma^5 v^{(s)}_{\mu\nu}u_\rho\mathrm{S}_\lambda + \beta_{21}\epsilon^{\mu\nu\rho\lambda}\gamma^5 v_\rho\mathrm{S}_\lambda[F^+_{\mu\nu}, v\cdot u]\\
&+\beta_{22}\gamma^5\mathrm{S}^\nu[\mathrm{D}^\mu, F^+_{\mu\nu}] + \beta_{23}\gamma^5\langle[v\cdot u, \chi_+]_+\rangle\\
&+\beta_2\gamma^5 v\cdot u\langle\chi_+\rangle + i\beta_{25}\gamma^5\langle[v\cdot\mathrm{D}, \chi_-]\rangle,
\end{aligned}
\tag{40}
$$

$\frac{1}{2\mathrm{m}}\Big[\gamma^0 B^{(3)\ \dagger}\gamma^0 B^{(1)} + \gamma^0 B^{(1)\ \dagger}\gamma^0 B^{(3)}\Big][v^{(s)}, F^+, \chi_-, \chi_+]$ and eliminating all terms proportional to the nonrelativistic eom by field redefinition of H, one gets:

$$
\begin{aligned}
&\frac{1}{2\mathrm{m}}\Bigg[\beta_4\Big(-2i[\mathrm{S}\cdot\mathrm{D}, [v\cdot u, \chi_+]_+]_+ - \frac{g_A^0}{2}[v\cdot u, [v\cdot u, \chi_+]_+]_+\Big)\\
&+\beta_{14}\Big(i[\mathrm{D}_\mu, [\chi_-, u^\mu]]_+ + g_A^0[\mathrm{S}\cdot u, [\chi_-, v\cdot u]]_+ - \epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[\mathrm{D}_\nu, [\chi_-, u_\mu]]\\
&-\frac{g_A^0}{2}[v\cdot u, [\chi_-, \mathrm{S}\cdot u]]_+\Big)\\
&+\Big(\frac{ig_A^0}{2}[\mathrm{D}^\mu, (i\beta_{17}v^{(s)}_{\mu\nu} + i\beta_{22}F^+_{\mu\nu})]v^\nu\mathrm{S}\cdot uu\\
&-\frac{1}{2}[\mathrm{D}^\mu, (i\beta_{17}v^{(s)}_{\mu\nu} + i\beta_{22}F^+_{\mu\nu})]\mathrm{D}_\nu\\
&+i\epsilon^{\nu\rho\lambda\sigma}v_\lambda\mathrm{S}_\sigma[\mathrm{D}^\mu, (i\beta_{17}v^{(s)}_{\mu\nu} + i\beta_{22}F^+_{\mu\nu})]\mathrm{D}^\rho\\
&-i\frac{g_A^0}{2}\mathrm{S}^\nu[\mathrm{D}^\mu, (i\beta_{17}v^{(s)}_{\mu\nu} + i\beta_{22}F^+_{\mu\nu})]v\cdot u\Big)\\
&+\beta_{18}\Big(-2v^\mu[F^+_{\mu\nu}, u^\nu]\mathrm{S}\cdot\mathrm{D} + \frac{ig_A^0}{2}v^\mu[F^+_{\mu\nu}, u^\nu]v\cdot u\Big) + \beta_{19}\Big(\frac{i}{2}\epsilon^{\lambda\alpha\rho\beta}[F^+_{\lambda\alpha}, u_\rho]_+(ig_A^0 v_\beta\mathrm{S}\cdot u - \mathrm{D}_\beta)\\
&-2([F^+_{\lambda\alpha}, u_\nu]_+v^\lambda\mathrm{S}^\alpha\mathrm{D}^\nu - \mathrm{S}^\alpha[F^+_{\lambda\alpha}, v\cdot u]_+\mathrm{D}^\lambda + v^\alpha[F^+_{\lambda\alpha}, \mathrm{S}\cdot u]_+\mathrm{D}^\nu)\\
&+\frac{g_A^0}{2}\epsilon^{\lambda\alpha\rho\beta}\mathrm{S}_\beta[F^+_{\lambda\alpha}, u_\rho]_+v\cdot u\Big) + \beta_{20}\Big(\frac{i}{2}\epsilon^{\lambda\alpha\rho\beta}v^{(s)}_{\lambda\alpha}u_\rho(-ig_A^0\mathrm{S}\cdot u - \mathrm{D}_\beta) + \frac{g_A^0}{2}\epsilon^{\lambda\alpha\rho\beta}v^{(s)}_{\lambda\alpha}u_\rho v\cdot u\\
&-2(v^{(s)}_{\lambda\alpha}v^\lambda\mathrm{S}^\alpha u_\nu\mathrm{D}^\nu - \mathrm{S}^\lambda v^{(s)}_{\lambda\alpha}v\cdot u\mathrm{D}^\lambda + v^\lambda v^{(s)}_{\lambda\alpha}\mathrm{S}\cdot u\mathrm{D}^\nu)\Big)\\
&+\beta_{21}\Big(\frac{i}{2}\epsilon^{\lambda\alpha\rho\beta}v_\rho[F^+_{\lambda\alpha}, v\cdot u]\mathrm{D}_\beta + 2[F^+_{\lambda\alpha}, v\cdot u]v^\lambda(-\frac{1}{4}u^\alpha + \frac{i}{2}\epsilon^{\mu\nu\rho\kappa}v_\rho\mathrm{S}_\kappa u_\mu)\\
&-2\mathrm{S}^\alpha[F^+_{\lambda\alpha}, v\cdot u]\mathrm{D}^\lambda - \frac{g_A^0}{2}\epsilon^{\lambda\alpha\rho\beta}v_\rho\mathrm{S}_\beta[F^+_{\lambda\alpha}, v\cdot u]v\cdot u\Big)\\
&+\beta_{23}\Big(2i\langle[v\cdot u, \chi_+]_+\rangle\mathrm{S}\cdot\mathrm{D} + \frac{g_A^0}{2}v\cdot u\langle[v\cdot u, \chi_+]_+\rangle\\
&+\beta_{24}(v\cdot u\langle\chi_+\rangle\mathrm{S}\cdot\mathrm{D} + \frac{g_A^0}{2}(v\cdot u)^2\langle\chi_+\rangle\Big)\\
&+\beta_{25}\Big(-2\langle[v\cdot\mathrm{D}, \chi_-\mathrm{S}\cdot\mathrm{D} + i\frac{g_A^0}{2}\langle[v\cdot\mathrm{D}, \chi_-]\rangle v\cdot u\Big)\Bigg].
\end{aligned}
\tag{41}
$$

The set $\{\beta_i\}$ can be related to the set $\{b_i\}$ of [6].

6 Conclusion

A complete list of O(q^4) terms for off-shell nucleons was obtained *working within HBChPT* using a phase rule obtained in [5], along with reductions from algebraic identities. We also obtain the on-shell O(q^4) terms, again within the framework of HBChPT. For off-shell nucleons, one gets a total of 106 $\mathcal{O}(q^4)$ terms (given in Tables 1 and 2). Of these 27 are finite. Contrary to what is claimed in [9], the earlier version of this paper itself contained (overcomplete) list of terms *including* external fields. Besides, the whole point of this paper (and that of [1] and [5]) is to carry out the 1/m-reduction without actually doing it; this is carried out by developing a method of imposing charge conjugation invariance directly within the nonrelativistic framework in terms of a phase rule (6) that can be used directly within HBChPT. Also, once having obtained the list of nonrelativistic terms, constructing their relativistic counterparts (for reasons given in [9]) can easily be done by following [5]. We are getting fewer terms than the ones given in [9]. Also, instead of working with symmetrized and anti-symmetrised commutators of D_μ and u_ν, we use (10) to eliminate $F^-_{\mu\nu}$ altogether.

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Table 1: The Allowed Terms

i	$(m,n,p,q,k,u.t)$	Terms	F($\equiv$Finite) D($\equiv$Divergent)$[d_i]$	E ON
1	(4,0,0,0,0,0,0)	$(v\cdot \mathrm{D})^4$	D$[d_{197}]$	E
2		D^4	F	E
3	(0,4,0,0,0,0,0)	$(v\cdot u)^4$	D$[d_5]$	ON
4		u^4	D$[d_2]$	ON
5		$[u_\mu, u_\nu]_+^2$	D$[d_1]$	ON
6		$[u_\mu, v\cdot u]_+^2$	D$[d_4]$	ON
7	(2,2,0,0,0,0,0)	$v\cdot \mathrm{D}(v\cdot u)^2 v\cdot \mathrm{D}$	D$[d_{146}]$	E
8		$[v\cdot \mathrm{D},(v\cdot u)]^2$	D$[d_{128}]$	ON
9		$[v\cdot \mathrm{D}, u_\mu]^2$	D$[d_{133}]$	ON
10		$[\mathrm{D}_\mu,(v\cdot u)]^2$	D$[d_{135}]$	ON
11		$v\cdot \mathrm{D}[v\cdot u, u^\mu]_+\mathrm{D}_\mu + \mathrm{h.c.}$	D$[d_{148}]]$	E
12		$[\mathrm{D}_\mu, u_\nu]^2$	D$[d_{136}]$	ON
13		$\mathrm{D}_\mu u^2 \mathrm{D}^\mu$	D$[d_{150}]$	E
14		$[v\cdot u,[[v\cdot \mathrm{D}, v\cdot u], v\cdot \mathrm{D}]]_+$	D$[d_{129}]$	ON
15		$[v\cdot \mathrm{D},[[v\cdot \mathrm{D}, v\cdot u], v\cdot u]]_+$	D$[d_{138}]$	E
16		$[u_\mu,[[v\cdot \mathrm{D}, u^\mu], v\cdot \mathrm{D}]]_+$	D$[d_{134}]$	ON
17		$[v\cdot \mathrm{D},[[v\cdot \mathrm{D}, u_\mu], u^\mu]]_+$	D$[d_{142}]$	E
18		$[v\cdot u,[[\mathrm{D}^\mu, v\cdot u], \mathrm{D}_\mu]]_+$	D$[d_{131}]$	ON
19		$[u^\nu,[[\mathrm{D}_\mu, u_\nu]_+, \mathrm{D}^\mu]]$	D$[d_{137}]$	ON
20		$[\mathrm{D}_\mu,[[\mathrm{D}^\mu, u_\nu], u^\nu]]_+$	D$[d_{143}]$	E
21		$[\mathrm{D}_\mu,[[\mathrm{D}_\nu, u^\mu], u^\nu]]_+$	D$[d_{145}]$	E
22		$[u_\mu,[[\mathrm{D}^\mu, v\cdot u], v\cdot \mathrm{D}]]_+$	D$[d_{130}]$	ON
23		$[v\cdot \mathrm{D},[[\mathrm{D}_\nu, v\cdot u], u^\nu]]_+$	D$[d_{144}]$	E
24		$[[\mathrm{D}_\mu, v\cdot u],[v\cdot \mathrm{D}, u^\mu]]_+$	D$[d_{132}]$	ON
25		$i[\mathrm{D}_\mu,[\tilde{\chi}_-, u^\mu]]_+$	D$[d_{123}]$	E
26		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[\mathrm{D}_\kappa, u_\mu],[\mathrm{D}^\kappa, u_\nu]]$	D$[d_{169}]$	ON
27		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[[\mathrm{D}^\kappa, u_\mu], u_\nu]_+]_+$	D$[d_{175}]$	E
28	(2,0,1,0,0,0,0)	$[\mathrm{D}_\mu,[\mathrm{D}^\mu,\chi_+]]$	F	E
29		$\mathrm{D}_\mu\chi_+\mathrm{D}^\mu$	F	E
30		$v\cdot \mathrm{D}\chi_+ \mathrm{v}\cdot \mathrm{D}$	D$[d_{159,161}]$	E
31		$[v\cdot \mathrm{D},[\mathrm{v}\cdot \mathrm{D},\chi_+]]$	D$[d_{156,157}]$	E
32		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\mu,[\mathrm{D}_\nu,\chi_+]_+]$	F	E
33	(0,2,1,0,0,0,0)	$u_\mu\chi_+u^\mu$	D$[d_{10,11,12}]$	ON
34		$u^2\chi_+$	D$[d_{10,11}]$	ON
35		$v\cdot u\chi_+ v\cdot u$	D$[d_{13,14,15}]$	ON
36		$(v\cdot u)^2\chi_+$	D$[d_{13,14}]$	ON

Table 1: continued

i	(m,n,p,q)	Terms	F($\equiv$Finite) D($\equiv$Divergent)$[d_i]$	E ON
37		$i\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[u_\mu,u_\nu],\chi_+]_+$	F	ON
38	(0,0,2,0,0,0,0)	χ_+^2	D$[d_{21}]$	ON
39	(0,0,0,2,0,0,0)	χ_-^2	F	ON
40	(0,0,1,0,1,0,0)	$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda F^+_{\mu\nu}\chi_+$	D$[d_{54,56}]$	ON
41	(0,0,0,0,2,0,0)	$(F^+_{\mu\nu})^2$	F	ON
42		$v^\kappa v^\sigma[F^+_{\mu\kappa},F^{+\mu}_\sigma]_+$	D$[d_{18}]$	ON
43		$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[F^+_{\mu\kappa},F^{+\kappa}_\nu]$	D$[d_{52}]$	ON
44	(2,0,0,1,0,0)	$i[\mathrm{D}^\mu,[F^+_{\mu\nu},\mathrm{D}^\nu]]_+$	D$[d_{155}]$	E
45		$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[F^+_{\mu\kappa},\mathrm{D}^\kappa],\mathrm{D}_\nu]$	D$[d_{187}]$	E
46		$iv^\mu[[F^+_{\mu\nu},v\cdot\mathrm{D}],\mathrm{D}^\nu]_+$	D$[d_{153}]$	E
47		$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[\mathrm{D}_\kappa,[F^+_{\mu\nu},\mathrm{D}^\kappa]]$	D$[d_{181}]$	E
48		$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda \mathrm{D}_\kappa F^+_{\mu\nu}\mathrm{D}^\kappa$	D$[d_{184}]$	E
49	(0,0,1,0,0,0,1)	$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda v^{(s)}_{\mu\nu}\chi_+$	D$[d_{55}]$	ON
50	(0,0,0,0,1,0,1)	$v^{(s)}_{\mu\nu}F^{+\mu\nu}$	D$[d_{17}]$	ON
51		$v^\kappa v^\sigma v^{(s)}_{\kappa\mu}F^{+\ \mu}_\sigma$	D$[d_{19}]$	ON
52	(0,2,0,0,1,0,0)	$i[u_\mu,[F^+_{\mu\nu},u_\nu]]_+$	F	ON
53		$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[F^+_{\mu\kappa},[u^\kappa,u_\nu]_+]_+$	D$[d_{42}]$	ON
54		$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[F^+_{\mu\kappa},u_\nu]_+,u^\kappa]_+$	D$[d_{41}]$	ON
55		$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda[[F^+_{\mu\kappa},u^\kappa]_+,u_\nu]_+$	D$[d_{40}]$	ON
56		$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda u_\kappa F^+_{\mu\nu}u^\kappa$	D$[d_{39}]$	ON
57		$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda F^+_{\mu\nu}u^2$	D$[d_{37}]$	ON
58		$iv^\mu[[\mathrm{D}^\nu,v^{(s)}_{\mu\nu}],v\cdot\mathrm{D}]_+$	D$[d_{151}$	E
59		$\epsilon^{\mu\nu\rho\lambda}v_\rho \mathrm{S}_\lambda v^{(s)}_{\mu\nu}u^2$	D$[d_{38}]$	ON
60	(3,1,0,0,0,0,0)	$i[v\cdot\mathrm{D},[v\cdot\mathrm{D},[v\cdot\mathrm{D},\mathrm{S}\cdot u]]]_+$	D$[d_{190}]$	E
61		$i[v\cdot u,\mathrm{S}\cdot u]_+v\cdot uv\cdot\mathrm{D}+$ h.c.	D$[d_{91}]$	E
62	(1,1,1,0,0,0,0)	$i[[v\cdot\mathrm{D},\mathrm{S}\cdot u]_+,\chi_+]_+$	D$[d_{117}]$	E
63		$i[[\mathrm{S}\cdot\mathrm{D},v\cdot u]_+,\chi$	D$[d_{118}]$	E
64	(1,1,0,0,1,0,1)	$v^{[\mu}\mathrm{S}^{\nu]}[F^+_{\mu\nu},v\cdot\mathrm{D}],v\cdot u]_+$	D$[d_{99}]$	ON
65		$v^{[\mu}\mathrm{S}^{\nu]}[F^+_{\mu\nu},[v\cdot\mathrm{D},v\cdot u]]_+$	D$[d_{101}]$	ON
66		$v^{[\mu}\mathrm{S}^{\nu]}[v\cdot\mathrm{D},[F^+_{\mu\nu},v\cdot u]]_+$	D$[d_{108}]$	E
67		$v^{[\mu}\mathrm{S}^{\nu]}[[v^{(s)}_{\mu\nu},v\cdot\mathrm{D}],v\cdot u]_+$	D$[d_{94}]$	ON
68		$v^{[\mu}\mathrm{S}^{\nu]}[v^{(s)}_{\mu\nu},[v\cdot\mathrm{D},v\cdot u]]_+$	D$[d_{98}]$	ON

Table 2: The terms that need to be included if assume isospin violation

i	$(m,n,p,q,k,u.t)$	Terms	F($\equiv$Finite) D($\equiv$Divergent)[d_i]	E ON
69	(2,0,0,,1,0,0,0)	$\langle[\mathrm{D}_\mu,[\mathrm{D}^\mu,\chi_+]]\rangle$	D[d_{158}]	ON
70		$\mathrm{D}_\mu\langle\chi_+\rangle\mathrm{D}^\mu$	D[d_{160}]	E
71		$\langle[v\cdot\mathrm{D},[v\cdot\mathrm{D},\chi_+]]\rangle$	D[d_{157}]	ON
72		$v\cdot\mathrm{D}\langle\chi_+\rangle v\cdot\mathrm{D}$	D[d_{159}]	E
73	(0,2,0,1,0,0,0)	$u^2\langle\chi_+\rangle$	D[d_{10}]	ON
74		$u^\mu\langle[u_\mu,\chi_+]_+\rangle$	D[d_{12}]	ON
75		$(v\cdot u)^2\langle\chi_+\rangle$	D[d_{13}]	ON
76		$v\cdot u\langle[v\cdot u,\chi_+]\rangle$	D[d_{15}]	ON
77		$i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda\langle[[u_\mu,u_\nu],\chi_+]_+\rangle$	D[d_{50}	ON
78		$i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[u_\mu,u_\nu]\langle\chi_+\rangle$	D[d_{51}]	ON
79		$i\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[u_\mu,[u_\nu,\chi]_+]_+$	F	ON
80	(0,0,1,0,1,0,0)	$\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda F^+_{\mu\nu}\langle\chi_+\rangle$	D[$d_{54,55}$]	ON
81	(0,0,1,0,0,1,0)	$\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda v^{(s)}_{\mu\nu}\langle\chi_+\rangle$	D[d_{55}]	ON
82	(0,0,0,2,0,0,0)	$\chi_+\langle\chi_+\rangle$	D[$d_{20,21}$]	ON
83		$\langle\chi_+\rangle^2$	D[d_{21}]	ON
84	(0,0,0,2,0,0,0)	$\chi_-\langle\chi_-\rangle$	F	ON
85		$\langle\chi_-\rangle^2$	F	ON
86	(1,1,0,1,0,0,0)	$i[\chi_+,[v\cdot\mathrm{D},\mathrm{S}\cdot u]]$	D[d_{115}]	ON
87		$i[[\chi_+,v\cdot\mathrm{D}],\mathrm{S}\cdot u]$	D[d_{116}]	ON
88		$i[\chi_+,[\mathrm{S}\cdot\mathrm{D},v\cdot u]]$	F	ON
89		$i[[\chi_+,\mathrm{S}\cdot\mathrm{D}],v\cdot u]$	F	ON
90		$i[v\cdot\mathrm{D},\langle[\mathrm{S}\cdot u,\chi_+]_+\rangle]_+$	D[d_{119}]	E
91		$i[\mathrm{S}\cdot\mathrm{D},\langle[v\cdot u,\chi_+]_+\rangle]_+$	F	E
92		$i\langle\chi_+\rangle[\mathrm{S}\cdot\mathrm{D},v\cdot u]_+$	D[d_{119}]	ON
93		$i[\langle\chi_+\rangle[v\cdot\mathrm{D},\mathrm{S}\cdot u]_+$	D[d_{117}]	ON
94	(1,1,0,0,1,0,0)	$i\langle\chi_-\rangle[\mathrm{D}_\mu,u^\mu]$	F	ON
95		$i[\mathrm{D}_\mu,[u^\mu,\langle\chi_-\rangle]_+]$	F	E
96		$i\langle\chi_-\rangle[v\cdot\mathrm{D},v\cdot u]$	F	ON
97		$i[v\cdot\mathrm{D},[v\cdot u,\langle\chi_-\rangle]_+]$	F	E
98		$\epsilon^{\mu\nu\rho\lambda}v_\rho\mathrm{S}_\lambda[\mathrm{D}_\mu,\langle[u_\nu,\chi_-]\rangle]_+$	F	E
99	(0,2,0,0,1,0,0,0)	$[[v\cdot u,\mathrm{S}\cdot u],\chi_-]_+$	F	ON
100		$[v\cdot u,[\mathrm{S}\cdot u,\chi_-]]_+$	F	ON
101		$[\mathrm{S}\cdot u,[v\cdot u,\chi_-]]_+$	F	ON
102		$[[v\cdot u,\mathrm{S}\cdot u]\langle\chi_-\rangle$	F	ON
103	(2,0,0,1,0,0,0)	$i[v\cdot\mathrm{D},[\mathrm{S}\cdot\mathrm{D},\langle\chi_-\rangle]_+]$	F	E
104	(2,0,0,1,0,0,0)	$i[v\cdot\mathrm{D},[\mathrm{S}\cdot\mathrm{D},\langle\chi_-\rangle]_+]$	F	E
105	(0,0,0,1,1,0,0)	$iv^{[\mu}\mathrm{S}^{\nu]}F^+_{\mu\nu}\langle\chi_-\rangle$	F	ON
106	(0,0,1,,0,0,1)	$iv^{[\mu}\mathrm{S}^{\nu]}v^{(s)}_{\mu\nu}\langle\chi_-\rangle$	F	ON